Solutions to Exercises - Lecture (1, 2) up to Ex. 6.

(II.8.1) From an earlier exercise we have that $Y = X_1 + \cdots + X_l \sim \text{binomial}(n, p_1 + \cdots + p_l)$. Therefore

$$p_{(X_1,\ldots,X_k)\mid Y}(x_1,\ldots,x_k \mid y) = \frac{\binom{n}{x_1\, x_2 \cdots x_k} p_1^{x_1} \cdots p_k^{x_k}}{\binom{n}{y}\left(\sum_{i=1}^{l} p_i\right)^{y}\left(1-\sum_{i=1}^{l} p_i\right)^{n-y}}$$

$$= \frac{y!}{x_1! \cdots x_l!}\left(\frac{p_1}{\sum_{i=1}^{l} p_i}\right)^{x_1} \cdots \left(\frac{p_l}{\sum_{i=1}^{l} p_i}\right)^{x_l} \times$$

$$\frac{(n-y)!}{(x_{l+1})! \cdots x_k!}\left(\frac{p_{l+1}}{\sum_{i=l+1}^{k} p_i}\right)^{x_{l+1}} \cdots \left(\frac{p_k}{\sum_{i=l+1}^{k} p_i}\right)^{x_k}$$

$$= \text{multinomial}\left(y, \frac{p_1}{\sum_{i=1}^{l} p_i}, \ldots, \frac{p_l}{\sum_{i=1}^{l} p_i}\right) \times$$

$$\text{multinomial}\left(n-y, \frac{p_{l+1}}{\sum_{i=l+1}^{k} p_i}, \ldots, \frac{p_k}{\sum_{i=l+1}^{k} p_i}\right)$$

(II.8.2) Since $T(x_1, \ldots, x_k) = x_1$, $J_T(x_1, \ldots, x_k)$

$$= \left|\det\left(\frac{\partial T}{\partial x_1}\ \frac{\partial T}{\partial x_2} \cdots \frac{\partial T}{\partial x_k}\right)\begin{pmatrix}\frac{\partial T}{\partial x_1}\\ \vdots \\ \frac{\partial T}{\partial x_k}\end{pmatrix}\right|^{-\frac{1}{2}}$$

$$\left|\det(1\ 0 \cdots 0)\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}\right|^{-\frac{1}{2}} = |\det(1)|^{-1/2} = 1 \text{ and}$$

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{(X_1,\ldots,X_k)}(x_1, z_2, \ldots, z_k)\, dz_2 \cdots dz_k$$

Therefore $f_{(x_2,\dots,x_n)|x_1}(x_2,\dots,x_n|x_1) = \dfrac{f_{(x_1,\dots,x_n)}(x_1,\dots,x_n)\, J_T(x_1,\dots,x_n)}{f_{x_1}(x_1)}$

$= f_{(x_1,\dots,x_n)}(x_1,\dots,x_n) / f_{x_1}(x_1)$.

11.8.3 We know

$$0 \le \underset{\sim}{x}' \Sigma \underset{\sim}{x} = \begin{pmatrix} \underset{\sim}{x}_1 \\ \underset{\sim}{x}_2 \end{pmatrix}' \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}' & \Sigma_{22} \end{pmatrix} \begin{pmatrix} \underset{\sim}{x}_1 \\ \underset{\sim}{x}_2 \end{pmatrix}$$

$$= \left(\underset{\sim}{x}_1' \Sigma_{11} + \underset{\sim}{x}_2' \Sigma_{12} \quad \underset{\sim}{x}_1' \Sigma_{12} + \underset{\sim}{x}_2' \Sigma_{22} \right) \begin{pmatrix} \underset{\sim}{x}_1 \\ \underset{\sim}{x}_2 \end{pmatrix}$$

$$= \underset{\sim}{x}_1' \Sigma_{11} \underset{\sim}{x}_1 + \underset{\sim}{x}_2' \Sigma_{12} \underset{\sim}{x}_1 + \underset{\sim}{x}_1' \Sigma_{12} \underset{\sim}{x}_2 + \underset{\sim}{x}_2' \Sigma_{22} \underset{\sim}{x}_2$$

$$= \underset{\sim}{x}_1' \Sigma_{11} \underset{\sim}{x}_1 + 2 \underset{\sim}{x}_1' \Sigma_{12} \underset{\sim}{x}_2 + \underset{\sim}{x}_2' \Sigma_{22} \underset{\sim}{x}_2 > 0$$

whenever $\underset{\sim}{x} \ne 0$. So putting $\underset{\sim}{x}_2 = \underset{\sim}{0}$ we have $\underset{\sim}{x}_1' \Sigma_{11} \underset{\sim}{x}_1 = \underset{\sim}{x}' \Sigma \underset{\sim}{x} > 0$ whenever $\underset{\sim}{x}_1 \ne \underset{\sim}{0}$ which implies Σ_{11} is p.d. Similarly Σ_{22} is p.d.

11.8.4 Suppose $A = QR_1 = QR_2$ where R_1, R_2 are upper Δ with positive diagonal. Then $Q'A = R_1 = R_2$.

II.8.5 (i) $\begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}$

$= \begin{pmatrix} a_1 a_2 & a_1 b_2 + b_1 c_2 \\ 0 & c_1 c_2 \end{pmatrix}$ is upper Δ and if

$a_1 > 0, c_1 > 0, a_2 > 0, c_2 > 0$ then $a_1 a_2 > 0, c_1 c_2 > 0$

(ii) $\det \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = ac - 0 \cdot b = ac > 0$ and

so $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ is nonsingular when $a > 0, c > 0$.

Also $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^{-1} = \frac{1}{ac} \begin{pmatrix} c & -b \\ 0 & a \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{ac} \\ 0 & \frac{1}{c} \end{pmatrix}$

since $\begin{pmatrix} \frac{1}{a} & -\frac{b}{ac} \\ 0 & \frac{1}{c} \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 1 & \frac{b}{a} - \frac{b}{a} \\ 0 & 1 \end{pmatrix} = I$.

(iii) Suppose $\Sigma = R_1' R_1 = R_2' R_2$ where R_1, R_2 are upper Δ with positive diagonal.

Then $\left[(R_1')^{-1} R_2' \right] R_2 R_1^{-1} = A' A = I$, so

$A^{-1} = (R_2 R_1^{-1})^{-1} = R_1 R_2^{-1} = (R_1')^{-1} R_2' = A'$.

But $R_1 R_2^{-1}$ is upper Δ while $(R_1')^{-1} R_2'$ is lower Δ so they are both diagonal as is $(R_1 R_2^{-1})^{-1} = R_2 R_1^{-1} = A$. But this implies that A is a diagonal orthogonal matrix with positive diagonal which implies $A = I$ which implies $R_2 R_1^{-1} = I$ or $R_1 = R_2$.

(iv)
$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ 0 & a_{22} & \cdots & a_{2k} \\ & & \ddots & \\ 0 & 0 & \cdots & a_{kk} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ 0 & b_{22} & \cdots & b_{2k} \\ & & \ddots & \\ 0 & 0 & \cdots & b_{kk} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} + a_{12}b_{22} & \cdots & a_{11}b_{1k} + \cdots + a_{1k}b_{kk} \\ 0 & a_{22}b_{22} & \cdots & a_{22}b_{2k} + \cdots + a_{2k}b_{kk} \\ \vdots & 0 & \ddots & \\ 0 & & & a_{kk}b_{kk} \end{pmatrix}$$

so the product of two upper Δ matrices with positive diagonals is upper Δ with positive diagonal.

$$\text{Also } \det A = \sum_{\sigma \in S_k} \operatorname{sgn}\sigma \prod_{i=1}^{k} a_{i\sigma_i}$$

where S_k is the set of all permutations of $(1, \ldots, k)$ and $\sigma = (\sigma_1, \ldots, \sigma_k) \in S_k$ and $\operatorname{sgn}\sigma = 1$ when $(\sigma_1, \ldots, \sigma_k)$ is obtained from $(1, \ldots, k)$ by an even number of pairwise switches (transpositions) and otherwise $\operatorname{sgn}\sigma = -1$. Note that $\prod_{i=1}^{k} a_{i\sigma_i}$ contains one element from each column and one element from each row of A in the product. When A is upper Δ the only product term that is nonzero is $a_{11} a_{22} \cdots a_{nn}$ and this is >0 when $a_{ii} > 0 \ \forall i$. Since $\sigma =$ identity in this case $\operatorname{sgn} \sigma = 1$ and so $\det A = a_{11} a_{22} \cdots a_{nn}$. Therefore A is upper Δ.

For $B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ 0 & b_{22} & & \\ & 0 & & 0 \end{pmatrix}$ to be the inverse of A we must have

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ 0 & a_{22} & \cdots & a_{2k} \\ \vdots & & & \\ 0 & \cdots & & a_{nn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ 0 & b_{22} & \cdots & b_{2k} \\ \vdots & & & \vdots \\ 0 & \cdots & & b_{kk} \end{pmatrix} = I \text{ so}$$

$a_{ii} b_{ii} = 1$ implying $b_{ii} = 1/a_{ii} \quad i = 1, \ldots, k$

then <u>backsolving</u>

$$0 = a_{k-1k-1} b_{k-1k} + a_{k-1k} b_{kk} = a_{k-1k-1} b_{k-1k} + \frac{a_{k-1k}}{a_{kk}}$$

$$\text{so } b_{k-1k} = -\frac{a_{k-1k}}{a_{k-1k-1}} b_{kk}$$

$$0 = a_{k-2k-2} b_{k-2k} + a_{k-2k-1} b_{k-1k} + a_{k-2k} b_{kk}$$

$$\text{or } b_{k-2k} = -\frac{a_{k-2k-1} b_{k-1k}}{a_{k-2k-2}} - \frac{a_{k-2k}}{a_{k-2k-2}} b_{kk}$$

and so we can determine all the entries of B in this way which proves A^{-1} is upper Δ with positive diagonal whenever A is.

The proof that the Cholesky factor is unique is as in (iii).

II.8.6 Suppose $(x_{i_1}, \ldots, x_{i_l})$ is a subvector of $\underset{\sim}{X} \sim N_k(\underset{\sim}{\mu}, \Sigma)$. Let $A \in \mathbb{R}^{k \times k}$ be the matrix with 1st row equal to the standard basis vector $\underset{\sim}{e}_{i_1}$ (1 in i_1'st place and 0's elsewhere), with 2nd row equal to the standard basis vector $\underset{\sim}{e}_{i_2}$, etc. and with the last $k-l$ rows equal to the remaining basis vectors in any order. Now let

$$\underset{\sim}{Y} = A\underset{\sim}{X} = \begin{matrix} l \\ k-l \end{matrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}\underset{\sim}{X} = \begin{pmatrix} A_1\underset{\sim}{X} \\ A_2\underset{\sim}{X} \end{pmatrix}$$

$$\text{where } A_1\underset{\sim}{X} = \begin{pmatrix} x_{i_1} \\ \vdots \\ x_{i_l} \end{pmatrix}$$

$$\sim N_k(A\underset{\sim}{\mu}, A\Sigma A') \text{ and,}$$

$$A\underset{\sim}{\mu} = \begin{pmatrix} A_1\underset{\sim}{\mu} \\ A_2\underset{\sim}{\mu} \end{pmatrix}, \quad A\Sigma A' = \begin{pmatrix} A_1\Sigma A_1' & A_1\Sigma A_2' \\ A_2\Sigma A_1' & A_2\Sigma A_2' \end{pmatrix}$$

Therefore $\underset{\sim}{Y}_i = \begin{pmatrix} x_{i_1} \\ \vdots \\ x_{i_\ell} \end{pmatrix} \sim N(A_i \underset{\sim}{\mu}, A_i \Sigma A_i')$

and $A_i \underset{\sim}{\mu} = \begin{pmatrix} \mu_{i_1} \\ \vdots \\ \mu_{i_\ell} \end{pmatrix}$, $A \Sigma A_i' = \begin{pmatrix} \sigma_{i_1 i_1} & \sigma_{i_1 i_2} & \cdots & \sigma_{i_1 i_\ell} \\ \vdots & & & \\ \sigma_{i_1 i_\ell} & \cdots & & \sigma_{i_\ell i_\ell} \end{pmatrix}$