Solutions to Exercises - Lecture 1

(I.1.1) A uniform prob. measure on the symmetrical cube's faces assigns prob. 1/6 to each face. Then by symmetry this implies $P(\{1\}) = 2/6 = 1/3$, $P(\{2\}) = 3/6 = 1/2$, $P(\{3\}) = 1/6$ for the labelled cube.

(I.1.2) We have $P(A_1 \cup A_2) = P(A_1) + P(A_2)$ by (i) on page 6. Assume for some fixed n that $P(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n P(A_i)$ when $A_1, \ldots, A_n$ are mutually disjoint. Now consider $A_1, \ldots, A_{n+1}$ mutually disjoint. Then

$$P\left(\bigcup_{i=1}^{n+1} A_i\right) = P\left(\left(\bigcup_{i=1}^{n} A_i\right) \cup A_{n+1}\right)$$

$= P\left(\bigcup_{i=1}^{n} A_i\right) + P(A_{n+1})$ since $A_{n+1} \cap \bigcup_{i=1}^{n} A_i = \phi$ and additivity holds with two sets

$= \left(\sum_{i=1}^{n} P(A_i)\right) + P(A_{n+1})$ induction hypothesis

$= \sum_{i=1}^{n+1} P(A_i)$ and so P is finitely additive.

(I.1.3) We have $A \cap A^c = \phi$ and $A \cup A^c = \underline{\Omega}$, so $1 = P(\underline{\Omega}) = P(A \cup A^c) = P(A) + P(A^c)$ by (i) p. 6 and therefore $P(A^c) = 1 - P(A)$.

(I.1.4) Write $A \cup B = (A \cap B^c) \cup (A \cap B) \cup (A^c \cap B)$ and note that $A \cap B^c$, $A \cap B$, $A^c \cap B$ are mutually disjoint. Also $A = (A \cap B^c) \cup (A \cap B)$ and $B = (A^c \cap B) \cup (A \cap B)$. Therefore, using the finite additivity of P,

$$P(A \cup B) = P(A \cap B^c) + P(A \cap B) + P(A^c \cap B)$$
$$= (P(A \cap B^c) + P(A \cap B)) + (P(A^c \cap B) + P(A \cap B)) - P(A \cap B)$$
$$= P(A) + P(B) - P(A \cap B)$$

I.1.6 For the initial context $\Omega = \{1,2,3,4\}$

$$2^{\Omega} = \{\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \Omega\}$$

$$P(\{1\}) = P(\{2\}) = P(\{3\}) = P(\{4\}) = \frac{1}{4}$$

and P of any other set is obtained using these probabilities via additivity.

For the second context $\Omega = \{1,2,3\}$ and

$$2^{\Omega} = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \Omega\}$$

$$P(\{1\}) = \frac{1}{4}, \quad P(\{2\}) = \frac{1}{4}, \quad P(\{3\}) = \frac{1}{2}$$

with all the other probabilities obtained from these using additivity.

I.2.1 Suppose $\omega \in \left(\bigcap_{i=1}^{\infty} A_i\right)^c$, then $\omega \notin \bigcap_{i=1}^{\infty} A_i$ and so $\omega \notin A_i$ for some i, which implies $\omega \in A_i^c$ which implies $\omega \in \bigcup_{i=1}^{\infty} A_i^c$.

Now suppose $\omega \in \bigcup_{i=1}^{\infty} A_i^c$. Then $\omega \in A_i^c$ for some i, which implies $\omega \notin A_i$ for some i, which implies $\omega \notin \bigcap_{i=1}^{\infty} A_i$, which implies $\omega \in \left(\bigcap_{i=1}^{\infty} A_i\right)^c$.

Since we have shown LHS $\subseteq$ RHS and RHS $\subseteq$ LHS this proves LHS $=$ RHS as required.

I.2.2 Since $A_i \in \mathcal{A}\ \forall i$, then $A_i^c \in \mathcal{A}\ \forall i$ which implies $\bigcup_{i=1}^{\infty} A_i^c \in \mathcal{A}$ which implies $\left(\bigcup_{i=1}^{\infty} A_i^c\right)^c \in \mathcal{A}$. Now, using Prop. 1.2.1 (i)

$$\left(\bigcup_{i=1}^{\infty} A_i^c\right)^c = \bigcap_{i=1}^{\infty} (A_i^c)^c = \bigcap_{i=1}^{\infty} A_i \quad \text{since } (A_i^c)^c = A_i.$$

Also $\Omega = \phi^c \in \mathcal{A}$ since $\phi \in \mathcal{A}$ and property (iii).

I.2.3 Let $A_{n+1} = A_{n+2} = \cdots = \phi$. Then $A_i \in \mathcal{A}\ \forall i$ and so $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$ but $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{n} A_i$.

Now let $A_{n+1} = A_{n+2} = \cdots = \Omega$. Then by Ex. 1.2.2

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{n} A_i \in \mathcal{A}.$$

I.3.1 Let $A_{n+1} = A_{n+2} = \cdots = \phi \in \mathcal{A}$ so $\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{\infty} A_i$ and so $P\left(\bigcup_{i=1}^{n} A_i\right) = P\left(\bigcup_{i=1}^{\infty} A_i\right)$

$= \sum_{i=1}^{\infty} P(A_i)$ since the A_i are mutually disjoint

$= \sum_{i=1}^{n} P(A_i)$ since $P(A_i) = 0$ for $i > n$.

(1.3.2)

[redacted] We have $B = A \cup (A^c \cap B)$ and A, $A^c \cap B$ are disjoint. Therefore $P(B) = P(A) + P(A^c \cap B) \geq P(A)$ since $P(A^c \cap B) \geq 0$.

I.3.3 [redacted] By Ex. 1.2.4 $1 = P(\Omega) = P(A \cup A^c) = P(A) + P(A^c)$ which gives the result.

I.3.4 [redacted] It is clear that $\mathcal{A} = \{\phi, \{1,2\}, \{3,4\}, \Omega\}$ is a σ-algebra on $\Omega = \{1,2,3,4\}$. Further $P(\Omega) = 1$ while $P(A \cup \phi) = P(A) + 0 = P(A) + P(\phi)$ for any $A \in \mathcal{A}$. $P(\{1,2\} \cup \{3,4\}) = P(\Omega) = 1 = 1/3 + 2/3 = P(\{1,2\}) + P(\{3,4\})$. Since there are no additional disjoint sets this proves P is normed and additive. That P is countably additive follows because if $A_1, A_2, \ldots \in \mathcal{A}$ are mutually disjoint then at most 2 of these sets equal $\{1,2\}$ or $\{3,4\}$ (at most once each) with all the remaining A_i equal to ϕ. Therefore

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \begin{cases} 0 & \text{all } A_i = \phi \\ P(\{1,2\}) & \text{one } A_i = \{1,2\} \text{ the rest} = \phi \\ P(\{3,4\}) & \text{" " " } \{3,4\} \text{ " " "} \\ 1 & \text{" " " } \{1,2\} \text{ and one } A_i = \{3,4\} \\ 1 & \text{one } A_i = \Omega \text{ the rest} = \phi \end{cases}$$

$$= \sum_{i=1}^{\infty} P(A_i)$$

Therefore P is countably additive and $(\Omega, \mathcal{A}, P)$ is a probability model.