Exercises Lecture 20 - Solutions

Ex VII.9.1 Consider $h : (\mathbb{R}^k, \mathcal{B}^k) \to (\mathbb{R}^1, \mathcal{B}^1)$. Then

$$E(Y h(\underset{\sim}{X})) = \int_{\mathbb{R}^{k+1}} y\, h(\underset{\sim}{x})\, f_{(\underset{\sim}{X},Y)}(\underset{\sim}{x},y)\, d(\underset{\sim}{x},y)$$

$$= \int_{\mathbb{R}^k}\left(\int_{-\infty}^{\infty} y\, h(\underset{\sim}{x})\, f_{(\underset{\sim}{X},Y)}(\underset{\sim}{x},y)\, dy\right) d\underset{\sim}{x}$$

$$= \int_{\mathbb{R}^k}\left(\int_{-\infty}^{\infty} y\, h(\underset{\sim}{x})\, f_{\underset{\sim}{X}}(\underset{\sim}{x})\, f_{Y|\underset{\sim}{X}}(y|\underset{\sim}{x})\, dy\right) d\underset{\sim}{x}$$

$$= \int_{\mathbb{R}^k} h(\underset{\sim}{x})\left(\int_{-\infty}^{\infty} y\, f_{Y|\underset{\sim}{X}}(y|\underset{\sim}{x})\, dy\right) f_{\underset{\sim}{X}}(\underset{\sim}{x})\, d\underset{\sim}{x}$$

since $h(\underset{\sim}{x})\, f_{\underset{\sim}{X}}(\underset{\sim}{x})$ is constant in the inner integral

$$= \int_{\mathbb{R}^k} h(\underset{\sim}{x})\, g(\underset{\sim}{x})\, f_{\underset{\sim}{X}}(\underset{\sim}{x})\, d\underset{\sim}{x}$$

where $g(\underset{\sim}{x}) = \int_{-\infty}^{\infty} y\, f_{Y|\underset{\sim}{X}}(y|\underset{\sim}{x})\, dy$

$= E(h(\underset{\sim}{X})\, g(\underset{\sim}{X}))$ and since this holds for all such h the general definition of conditional expectation implies

$g(\underset{\sim}{x}) = E(Y|\underset{\sim}{X})(\underset{\sim}{x})$.