Exercises Lecture 21 - Solutions

Ex III.9.1 $G_X(t) = E(t^X) = (1-p)t^0 + pt^1$
$= 1-p+pt$

When $X_1, \ldots, X_n \overset{iid}{\sim}$ Bernoulli(p) then $Y = X_1 + \cdots + X_n \sim$ binomial(n,p) so

$$G_Y(t) = G_X^n(t) = (1-p+pt)^n.$$

Ex. III.9.2 We use the definition $X \sim$ geometric(p) when X is the number of 0's before the first 1 so $P(X=n) = (1-p)^n p$. Then

$$G_X(t) = E(t^X) = \sum_{x=0}^{\infty} t^x (1-p)^x p$$

$$= p\sum_{x=0}^{\infty} (t(1-p))^x = \frac{p}{1-t(1-p)}$$

$$\mu_X = \frac{d\,G_X(t)}{dt}\Big|_{t=1} = \frac{(1-p)p}{(1-t(1-p))^2}\Big|_{t=1} = \frac{(1-p)p}{p^2} = \frac{1-p}{p}$$

$$\mu_{X[2]} = \frac{d^2 G_X(t)}{dt^2}\Big|_{t=1} = \frac{2(1-p)^2 p}{(1-t(1-p))^3}\Big|_{t=1} = \frac{2(1-p)^2 p}{p^3} = \frac{2(1-p)^2}{p^2}$$

Therefore $\sigma_X^2 = \mu_{X[2]} - \mu_X(\mu_X - 1)$

$$= \frac{2(1-p)^2}{p^2} - \frac{1-p}{p}\left(\frac{1-p}{p} - 1\right) = \frac{(1-p)^2}{p^2} + \frac{1-p}{p} = \frac{1-p}{p}\left(\frac{1-p}{p} + 1\right)$$

$$= \frac{1-p}{p^2}.$$

Ex. III.9.3 $G_N(t) = e^{\lambda(t-1)}$ and

$G_X(t) = E(t^X) = (1-p)t^{-1} + pt^1$

so $G_Y(t) = G_N(G_X(t)) = \exp\{\lambda(\frac{1-p}{t} + pt) - \lambda\}$

so $E(Y) = \left.\frac{dG_Y(t)}{dt}\right|_{t=1}$

$= \exp\{\lambda(\frac{(1-p)}{t} + pt) - \lambda\}\lambda(p - \frac{1-p}{t^2})\Big|_{t=1}$

$= \exp\{\lambda(1-p) + \lambda p - \lambda\}\lambda(p-1+p)$

$= \lambda(2p-1).$

Ex. III.9.4 Suppose $E(|X_1|^{j_1}|X_2|^{j_2}) < \infty$ $\forall j_1, j_2$ s.t. $j_1 + j_2 = j$. Then $E((|X_1| + |X_2|)^j) = \sum_{j_1+j_2=j} \binom{j}{j_1\, j_2} E(|X_1|^{j_1}|X_2|^{j_2}) < \infty$ and so by Prop III.3.4 $E((|X_1| + |X_2|)^i) < \infty$ $\forall i \le j$ and this implies $E(|X_1|^{i_1}|X_2|^{i_2}) < \infty$ $\forall i_1 \le j_1$ and $i_2 \le j_2$.

Note The original statement of Prop III.9.4 was not correct

Ex III.9.4 $m_{\underset{\sim}{y}}(\underset{\sim}{t}) = E(\exp\{\underset{\sim}{t}'\underset{\sim}{y}\})$

$= E(\exp\{\underset{\sim}{t}'\underset{\sim}{a} + \sum_{i=1}^{n} \underset{\sim}{t}' C_i \underset{\sim}{x}_i\})$

$= E(\exp\{\underset{\sim}{t}'\underset{\sim}{a}\} \prod_{i=1}^{n} \exp\{(C_i'\underset{\sim}{t})'\underset{\sim}{x}_i\})$

ind.
$= \exp\{\underset{\sim}{t}'\underset{\sim}{a}\} \prod_{i=1}^{n} E(\exp\{(C_i'\underset{\sim}{t})'\underset{\sim}{x}_i\})$

$= \exp\{\underset{\sim}{t}'\underset{\sim}{a}\} \prod_{i=1}^{n} m_{\underset{\sim}{x}_i}(C_i'\underset{\sim}{t})$

$= \exp\{\underset{\sim}{t}'\underset{\sim}{a}\} \prod_{i=1}^{n} \exp\{\underset{\sim}{t}' C_i \underset{\sim}{\mu}_i + \underset{\sim}{t}' C_i \Sigma_i C_i' \underset{\sim}{t}/2\}$

$= \exp\{\underset{\sim}{t}'(\underset{\sim}{a} + \sum_{i=1}^{n} C_i \underset{\sim}{\mu}_i) + \underset{\sim}{t}'(\sum_{i=1}^{n} C_i \Sigma_i C_i')\underset{\sim}{t}/2\}$

Thus by the Uniqueness Theorem

$\underset{\sim}{y} \sim N_m(\underset{\sim}{a} + \sum_{i=1}^{n} C_i \underset{\sim}{\mu}_i, \sum_{i=1}^{n} C_i \Sigma_i C_i')$.

Ex III.9.5 ETR 3.4.13

For geometric (θ) $m_x(t) = \sum_{i=0}^{\infty} e^{ti}(1-\theta)^i \theta$

$= \theta \sum_{i=0}^{\infty} (e^t(1-\theta))^i = \theta(1 - e^t(1-\theta))^{-1}$

when $|e^t(1-\theta)| < 1$ or $t < -\log(1-\theta)$

If $x_1, \ldots, x_r \overset{iid}{\sim}$ geometric(θ), then $x = \sum_{i=1}^{r} x_i$

$\sim$ negative binomial (r, θ). Therefore

$$m_X(t) = \prod_{i=1}^{r} m_{X_i}(t) = m_{X_1}^{r}(t)$$

$$= \theta^r \left(1 - e^t(1-\theta)\right)^{-r}$$

$$\left.\frac{d\, m_X(t)}{dt}\right|_{t=0} = \theta^r \left(r e^t (1-\theta)\left(1 - e^t(1-\theta)\right)^{-r-1} \right)\Big|_{t=0}$$

$$= r\theta^r (1-\theta)\, \theta^{-(r+1)} = r\,\frac{1-\theta}{\theta}$$

$$\left.\frac{d^2 m_X(t)}{dt^2}\right|_{t=0} = \theta^r \Big[r e^t (1-\theta)\left(1 - e^t(1-\theta)\right)^{-(r+1)}$$
$$+ r(r+1) e^{2t} (1-\theta)^2 \left(1 - e^t(1-\theta)\right)^{-(r+2)} \Big]\Big|_{t=0}$$

$$= \theta^r \left[r\, \frac{1-\theta}{\theta^{r+1}} + r(r+1)\frac{(1-\theta)^2}{\theta^{r+2}} \right]$$

$$= r\frac{(1-\theta)}{\theta}\left[1 + (r+1)\frac{(1-\theta)}{\theta} \right]$$

Therefore $\quad Var(X) = \dfrac{r(1-\theta)}{\theta}\left[1 + (r+1)\dfrac{(1-\theta)}{\theta} - r\dfrac{(1-\theta)}{\theta} \right]$

$$= \frac{r(1-\theta)}{\theta}\left[1 + \frac{1-\theta}{\theta} \right]$$

$$= \frac{r(1-\theta)}{\theta^2}$$

Ex III.9.6 EYR 3.4.16

$$m_Y(t) = E(e^{tY}) = \int_{-\infty}^{+\infty} e^{tx} \frac{e^{-|x|}}{2}\,dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{0} e^{x(1+t)}\,dx + \int_{0}^{+\infty} e^{-x(1-t)}\,dx\right]$$

$$= \frac{1}{2}\left[(1+t)^{-1} e^{x(1+t)}\Big|_{x=-\infty}^{0} + (1-t)^{-1} e^{-x(1-t)}\Big|_{x=0}^{\infty}\right]$$

$$= \frac{1}{2}\left[\frac{1}{1+t} + \frac{1}{1-t}\right] = \frac{1}{2}\,\frac{2}{(1+t)(1-t)} = \frac{1}{1-t^2}$$

for $|t| < 1$.

$$\frac{d\,m_Y(t)}{dt}\Big|_{t=0} = 2t(1-t^2)^{-2}\Big|_{t=0} = 0$$

$$\frac{d^2 m_Y(t)}{dt^2} = \left[2(1-t^2)^{-2} + 8t^2(1-t^2)^{-3}\right]\Big|_{t=0}$$

$$= 2$$

Therefore $Var(Y) = 2$.

Ex III.9.7 E+R 3.4.20

Suppose $X \sim \text{gamma}(\alpha, \lambda)$. Then

$$m_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\lambda-t)x} dx$$

$$= \lambda^\alpha (\lambda-t)^{-\alpha} \underbrace{\int_0^\infty \frac{(\lambda-t)^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\lambda-t)x} dx}_{=1}$$

since this is the integral of a gamma$(\alpha, \lambda-t)$ density provided $t < \lambda$

$$= (1 - t/\lambda)^{-\alpha} \quad \text{when } |t| < \lambda.$$

Ex. III.9.8 E+R 3.4.29

When $X \sim N(\mu, \sigma^2)$ then

$$c_X(t) = \exp\{i\mu t - \sigma^2 t^2/2\} \quad \text{so}$$

$\ln c_X(t) = i\mu t - \sigma^2 t^2/2$ and kth cumulant is

$$\left.\frac{d^k \ln c_X(t)}{i^k\, dt^k}\right|_{t=0} = \begin{cases} \mu & k=1 \\ \sigma^2 & k=2 \\ 0 & k>2 \end{cases}$$