Exercises - Lecture 23 - Solutions

Ex IV.3.1

(i) $E((a+bX+cY)^2) = E(a^2+b^2X^2+c^2Y^2 + 2abX + 2acY + 2bcXY)$

$= a^2 + b^2E(X^2) + c^2E(Y^2) + 2abE(X) + 2acE(Y) + 2bcE(XY)$

$< \infty$ since $E(X^2)<\infty$, $E(Y^2)<\infty$ implies $E(X), E(Y), E(XY) \in \mathbb{R}^1$.

(ii) $\langle aX+bY, Z\rangle = E((aX+bY)Z)$

$= E(aXZ+bYZ) = aE(XZ)+bE(YZ)$
and $E(XZ)\in\mathbb{R}^1$, $E(YZ)\in\mathbb{R}^1$ since $E(X^2)<\infty$, $E(Y^2)<\infty$, $E(Z^2)<\infty$

$= a\langle X,Z\rangle + b\langle Y,Z\rangle$

$\langle X,Y\rangle = E(XY) = \langle Y,X\rangle$

$\langle X,X\rangle = E(X^2) \geq 0$ and 0 only when $X \overset{wp1}{=} 0$

Therefore $\langle\cdot,\cdot\rangle$ has the three properties necessary to be an inner product.

(iii) $0 \leq \|X+Y\|^2 = E((X+Y)^2) = E(X^2+2XY+Y^2)$
$= E(X^2) + 2E(XY) + E(Y^2)$

Cauchy-Schwartz $\leq \|X\|^2 + 2\|X\|\,\|Y\| + \|Y\|^2$
$= (\|X\| + \|Y\|)^2$

and so $\|X+Y\| \leq \|X\| + \|Y\|$

$\|aX\| = (E(a^2X^2))^{1/2} = |a|\,\|X\|$

$\|X\| = 0$ iff $E(X^2) = 0$ iff $X \overset{wp1}{=} 0$.

Therefore $\|\cdot\|$ has the three properties of a norm.