Solutions - Lecture 3

(1.4.5) Let $\underset{\sim}{c}, \underset{\sim}{d} \in \mathbb{R}^k$ be s.t. $a_i \le c_i \le d_i \le b_i$ for $i = 1, \ldots, k$. Then define

$$P((\underset{\sim}{c}, \underset{\sim}{d}]) = \frac{\text{volume}((\underset{\sim}{c}, \underset{\sim}{d}])}{\text{volume}(\underset{\sim}{a}, \underset{\sim}{b}]}$$

$$= \prod_{i=1}^{k} (d_i - c_i) \Big/ \prod_{i=1}^{k} (b_i - a_i)$$

(1.4.1) Since $A_1 \subseteq A_2 \subseteq \ldots$ we have that

$\bigcap_{i=n}^{\infty} A_i = A_n$ so $\liminf A_n = \bigcup_{n=1}^{\infty} A_n$

Also $\bigcup_{i=n}^{\infty} A_i = \bigcup_{i=1}^{\infty} A_i$ and so

$\limsup A_n = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} A_i$.

Since $\liminf A_n = \limsup A_n = \bigcup_{i=1}^{\infty} A_i$ this proves

$A_n \to \bigcup_{i=1}^{\infty} A_i$.

(1.4.2) Since $A_1 \supseteq A_2 \supseteq \ldots$ we have that $A_1^c \subseteq A_2^c \subseteq \ldots$ is monotone increasing and so by the proof of Prop. 1.4.2

$$\lim_{n \to \infty} P(A_n^c) = P(\lim_{n \to \infty} A_n^c) = P(\bigcup_{i=1}^{\infty} A_i^c)$$

But $P(A_n^c) = 1 - P(A_n)$ and $P(\bigcup_{i=1}^{\infty} A_i^c)$

$= P((\bigcap_{i=1}^{\infty} A_i)^c) = 1 - P(\bigcap_{i=1}^{\infty} A_i)$

and so $\lim_{n\to\infty} P(A_n) = 1 - \lim P(A_n^c)$

$$= 1 - P\left(\bigcup_{i=1}^{\infty} A_i^c\right) = P\left(\bigcap_{i=1}^{\infty} A_i\right) = P\left(\lim_{n\to\infty} A_n\right).$$

1.6.1 (i) $P(\Omega \mid C) = P(\Omega \cap C)/P(C) = P(C)/P(C) = 1$

(ii) If $A_1, A_2, \dots \in \mathcal{A}$ are mutually disjoint

then $P\left(\bigcup_{i=1}^{\infty} A_i \mid C\right) = \dfrac{P\left(\left(\bigcup_{i=1}^{\infty} A_i\right) \cap C\right)}{P(C)}$

$= \dfrac{P\left(\bigcup_{i=1}^{\infty} (A_i \cap C)\right)}{P(C)}$ since $\left(\bigcup_{i=1}^{\infty} A_i\right) \cap C = \bigcup_{i=1}^{\infty} A_i \cap C$

$= \dfrac{\sum_{i=1}^{\infty} P(A_i \cap C)}{P(C)}$ since $(A_i \cap C) \cap (A_j \cap C) = \phi$ and using the fact that P is a prob. measure

$= \sum_{i=1}^{\infty} P(A_i \mid C).$

Therefore $(\Omega, \mathcal{A}, P(\cdot \mid C))$ is a probability model.

1.6.2 Note since $C_i \cap C_j = \phi$ when $i \neq j$ we have $(A \cap C_i) \cap (A \cap C_j) = \phi$ and $A = \bigcup_{i=1}^{\infty} A \cap C_i = A \cap \bigcup_{i=1}^{\infty} C_i$ and $\bigcup_{i=1}^{\infty} C_i = \Omega$.