Solutions to Exercises - Lecture 6

II.2.1 $\Delta^{(1)}_{a_1,b_1}\Delta^{(2)}_{a_2,b_2}F_{\underset{\sim}{X}} = \Delta^{(1)}_{a_1,b_1}\left(F_{\underset{\sim}{X}}(\cdot,b_2) - F_{\underset{\sim}{X}}(\cdot,a_2)\right)$

$= F_{\underset{\sim}{X}}(b_1,b_2) - F_{\underset{\sim}{X}}(b_1,a_2) - F_{\underset{\sim}{X}}(a_1,b_2) + F_{\underset{\sim}{X}}(a_1,a_2)$

$= \left(F_{\underset{\sim}{X}}(b_1,b_2) - F_{\underset{\sim}{X}}(a_1,b_2)\right) - \left(F_{\underset{\sim}{X}}(b_1,a_2) - F_{\underset{\sim}{X}}(a_1,a_2)\right)$

$= \Delta^{(2)}_{a_2,b_2}\Delta^{(1)}_{a_1,b_1}F_{\underset{\sim}{X}}$

$\Delta^{(1)}_{a_1,b_1}\Delta^{(2)}_{a_2,b_2}\Delta^{(3)}_{a_3,b_3}F_{\underset{\sim}{X}}$

$= \Delta^{(1)}_{a_1,b_1}\Delta^{(2)}_{a_2,b_2}\left(F_{\underset{\sim}{X}}(\cdot,\cdot,b_3) - F_{X}(\cdot,\cdot,a_3)\right)$

$= \Delta^{(1)}_{a_1,b_1}\Big(\left(F_{\underset{\sim}{X}}(\cdot,b_2,b_3) - F_{\underset{\sim}{X}}(\cdot,b_2,a_3)\right) -$

$\left(F_{\underset{\sim}{X}}(\cdot,a_2,b_3) - F_{\underset{\sim}{X}}(\cdot,a_2,a_3)\right)\Big)$

$= \left[F_{\underset{\sim}{X}}(b_1,b_2,b_3) - F_{\underset{\sim}{X}}(b_1,b_2,a_3) - F_{\underset{\sim}{X}}(b_1,a_2,b_3) + F_{\underset{\sim}{X}}(b_1,a_2,a_3)\right]$

$- \left[F_{\underset{\sim}{X}}(a_1,b_2,b_3) - F_{\underset{\sim}{X}}(a_1,b_2,a_3) - F_{\underset{\sim}{X}}(a_1,a_2,b_3) + F_{\underset{\sim}{X}}(a_1,a_2,a_3)\right]$

II.2.2 $\lim_{x_1\uparrow\infty,\dots,x_n\uparrow\infty} F_{\underset{\sim}{X}}(x_1,\dots,x_n)$

$= \lim_{x_1\uparrow\infty,\dots,x_n\uparrow\infty} P_{\underset{\sim}{X}}\left((-\infty,x_1]\times(-\infty,x_2]\times\dots\times(-\infty,x_n]\right)$

$= P_{\underset{\sim}{X}}(\mathbb{R}^k) = 1$ since $(-\infty,x_1]\times\dots\times(-\infty,x_n] \uparrow \mathbb{R}^k$

and we use the continuity of $P_{\underset{\sim}{X}}$.

VI.2.3 $\Delta^{(1)}_{a_1,b_1}\Delta^{(2)}_{a_2,b_2}F($

$= F(b_1,b_2) - F(b_1,a_2) - F(a_1,b_2) + F(a_1,a_2)$

and assuming $0 \le a_1 \le b_1$, $0 \le a_2 \le b_2$

$= (\cancel{1} - \cancel{e^{-b_1}} - \cancel{e^{-b_2}} + e^{-b_1-b_2}) - (\cancel{1} - \cancel{e^{-b_1}} - e^{-a_2} + e^{-b_1-a_2})$

$- (\cancel{1} - \cancel{e^{-a_1}} - \cancel{e^{-b_2}} + e^{-a_1-b_2}) + (\cancel{1} - \cancel{e^{-a_1}} - \cancel{e^{-a_2}} + e^{-a_1-a_2})$

$= e^{-b_1-b_2} - e^{-b_1-a_2} - e^{-a_1-b_2} + e^{-a_1-a_2}$

$= e^{-b_1}(e^{-b_2} - e^{-a_2}) - e^{-a_1}(e^{-b_2} - e^{-a_2})$

$= (e^{-b_1} - e^{-a_1})(e^{-b_2} - e^{-a_2})$

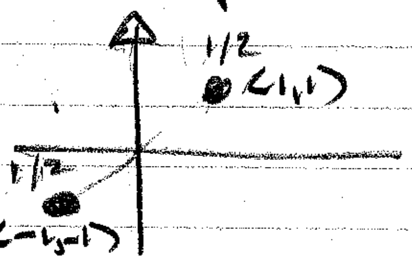
VI.2.4 (i) $P(\mathbb{R}^2) = 1$ since $(1,1), (-1,-1) \in \mathbb{R}^2$

(ii) Suppose $B_1, B_2, \ldots$ are mutually disjoint Borel sets. Suppose neither $(1,1)$ nor $(-1,-1)$ is in any of the B_i then neither point is in $\bigcup_{i=1}^{\infty} B_i$ and so $0 = P(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} P(B_i)$.

Suppose $(1,1) \in B_{i_1}$. Then no other B_i contains this point as the sets are mut. disjoint. If $(-1,-1)$ is not in any of the B_i then $(1,1) \in \bigcup_{i=1}^{\infty} B_i$, $(-1,-1) \notin \bigcup_{i=1}^{\infty} B_i$ and so

$\frac{1}{2} = P(\bigcup_{i=1}^{\infty} B_i) = P(B_{i_1}) = \sum_{i=1}^{\infty} P(B_i)$

The other cases where $(-1,-1)$ is in one of the B_i and $(1,1)$ isn't or both $(-1,-1) \in B_{i_1}$ and $(1,1) \in B_{i_2}$ for some i_1, i_2 can be dealt with similarly. Finally $P(\mathbb{R}^2) = 1$ since $(1,1), (-1,-1) \in \mathbb{R}^2$. This proves P is a prob. measure.



$$F(x_1,x_2) = \begin{cases} 0 & x_1 < -1 \text{ or } x_2 < -1 \\ 1/2 & -1 \le x_1 < 1,\ -1 \le x_2 < \infty \\ & \text{or } -1 \le x_1 < \infty,\ -1 \le x_2 < 1 \\ 1 & x_1 \ge 1,\ x_2 \ge 1 \end{cases}$$

IV.3.1 The number of possible samples of n with a_i labelled i and $a_i \in \{0,\dots,N_i\}$, $a_1 + \cdots + a_k = n$ is, by the multiplication principle,

(# of ways of choosing a_1 from N_1) $\times$ (# of ways of choosing a_2 from N_2)

$\times \cdots \times$ (# of ways of choosing a_k from N_k)

$= \binom{N_1}{a_1}\binom{N_2}{a_2}\cdots\binom{N_k}{a_k}$ and this gives the required probability.

For the numerical content we need to determine all the possible samples, namely, those that satisfy $a_1 \in \{0,1,2,3\}$, $a_2 \in \{0,1,2,3\}$, $a_3 \in \{0,1,2\}$ and $a_1 + a_2 + a_3 = 4$. The following (a_1, a_2, a_3) satisfy these constraints.

(3, 1, 0), (3, 0, 1)
(2, 2, 0), (2, 0, 2), (2, 1, 1)
(1, 3, 0), (1, 2, 1), (1, 1, 2),
(0, 3, 1), (0, 2, 2),