Solutions to Exercises – Lecture 9

IV.5.4 From Example IV.5.5 we have that

$$f_{\underset{\sim}{y}}(\underset{\sim}{y}) = f_{\underset{\sim}{x}}(A^{-1}(\underset{\sim}{y}-\underset{\sim}{b}))\,|\det A|^{-1}$$

$$= (2\pi)^{-k/2}(\det \Sigma)^{-\frac{1}{2}}(\det AA')^{-\frac{1}{2}} \times$$

$$\exp\{-\tfrac{1}{2}(A^{-1}(\underset{\sim}{y}-\underset{\sim}{b})-\underset{\sim}{\mu})'\Sigma^{-1}(A^{-1}(\underset{\sim}{y}-\underset{\sim}{b})-\underset{\sim}{\mu})\}$$

$$= (2\pi)^{-k/2}(\det A\Sigma A')^{-\frac{1}{2}} \times$$

$$\exp\{-\tfrac{1}{2}(\underset{\sim}{y}-A\underset{\sim}{\mu}-\underset{\sim}{b})'(A\Sigma A')^{-1}(\underset{\sim}{y}-A\underset{\sim}{\mu}-\underset{\sim}{b})\}$$

which says $\underset{\sim}{Y} \sim N_k(A\underset{\sim}{\mu}+\underset{\sim}{b},\, A\Sigma A')$ as required.

IV.5.5 By Example IV.5.5 and noting that

$\underset{\sim}{z} = C^{-1}\underset{\sim}{x} + (-C^{-1}\underset{\sim}{\mu})$ we have that

$$f_{\underset{\sim}{z}}(\underset{\sim}{z}) = f_{\underset{\sim}{x}}(C\underset{\sim}{z}+\underset{\sim}{\mu})\,|\det C^{-1}|^{-1}$$

$$= (2\pi)^{-k/2}(\det \Sigma)^{-\frac{1}{2}}(\det C^{-1}(C^{-1})')^{-\frac{1}{2}}$$

$$\exp\{-\tfrac{1}{2}(C\underset{\sim}{z}+\underset{\sim}{\mu}-\underset{\sim}{\mu})'\Sigma^{-1}(C\underset{\sim}{z}+\underset{\sim}{\mu}-\underset{\sim}{\mu})\}$$

$$= (2\pi)^{-k/2}\det(C^{-1}CC'(C')^{-1})^{-\frac{1}{2}}$$

$$\exp\{-\tfrac{1}{2}\underset{\sim}{z}'C'(CC')^{-1}C\underset{\sim}{z}\}$$

$$= (2\pi)^{-k/2} (\det I)^{-\frac{1}{2}} \exp\{-\tfrac{1}{2} \underset{\sim}{z}' I \underset{\sim}{z}\}$$

$$= (2\pi)^{-k/2} \exp\{-\tfrac{1}{2}\underset{\sim}{z}'\underset{\sim}{z}\}$$ and so $\underset{\sim}{z} \sim N_k(\underset{\sim}{0}, I)$.

II.5.6 Since $\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$ we have that

$\Sigma^{-1} = \dfrac{1}{\sigma_{11}\sigma_{22} - \sigma_{12}^2} \begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{pmatrix}$. Therefore

$(\det \Sigma)^{-1/2} = (\sigma_{11}\sigma_{22} - \sigma_{12}^2)^{-1/2}$, $(\underset{\sim}{x} - \underset{\sim}{\mu})' \Sigma^{-1} (\underset{\sim}{x} - \underset{\sim}{\mu})$

$$= \frac{1}{\sigma_{11}\sigma_{22} - \sigma_{12}^2} \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix}' \begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{pmatrix} \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix}$$

$$= \frac{1}{\sigma_{11}\sigma_{22} - \sigma_{12}^2} \left[(\sigma_{22}(x_1-\mu_1) - \sigma_{12}(x_2-\mu_2), -\sigma_{12}(x_1-\mu_1) + \sigma_{11}(x_2-\mu_2)) \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix} \right]$$

$$= \frac{1}{\sigma_{11}\sigma_{22} - \sigma_{12}^2} \left[\sigma_{22}(x_1-\mu_1)^2 - 2\sigma_{12}(x_1-\mu_1)(x_2-\mu_2) + \sigma_{11}(x_2-\mu_2)^2 \right]$$

$$= \frac{\sigma_{11}\sigma_{22}}{\sigma_{11}\sigma_{22} - \sigma_{12}^2} \left[\frac{(x_1-\mu_1)^2}{\sigma_{11}} - \frac{2\sigma_{12}}{\sigma_{11}\sigma_{22}}(x_1-\mu_1)(x_2-\mu_2) + \frac{(x_2-\mu_2)^2}{\sigma_{22}} \right]$$

and putting $\sigma_1 = \sqrt{\sigma_{11}}$, $\sigma_2 = \sqrt{\sigma_{22}}$
$\rho = \sigma_{12}/\sigma_1\sigma_2$

$$= (1-\rho^2)^{-1} \left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 \right]$$

Therefore

$$f_{\underset{\sim}{x}}(\underset{\sim}{x}) = \frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}} \exp\left\{ \frac{-1}{2(1-\rho^2)} [\quad] \right\}$$

VI.5.7 $\Sigma(Q\Lambda^{-1}Q') = Q\Lambda Q' Q\Lambda^{-1}Q'$

$= Q\Lambda I \Lambda^{-1}Q' = Q\Lambda\Lambda^{-1}Q' = QIQ'$

$= QQ' = I$. Therefore $\Sigma^{-1} = Q\Lambda^{-1}Q'$