Probability and Stochastic Processes I - Lecture 24

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- $\{(t, X_t) : t \in T\}$ is a Gaussian process if for any $\{t_1, \ldots., t_n\} \subset T$, then

$$\begin{pmatrix} X_{t_1} \\ \vdots \\ X_{t_n} \end{pmatrix} \sim N_n\left(\begin{pmatrix} \mu(t_1) \\ \vdots \\ \mu(t_n) \end{pmatrix}, \begin{pmatrix} \sigma(t_1, t_1) & \cdots & \sigma(t_1, t_n) \\ \vdots & & \vdots \\ \sigma(t_n, t_1) & \cdots & \sigma(t_n, t_n) \end{pmatrix}\right)$$

for some mean function $\mu : T \to R^1$ and autocovariance function $\sigma : T \times T \to R^1$

- when $T \subset R^1$ then a weakly stationary Gaussian process has μ constant and $\sigma(t_i, t_j) = \kappa(t_i - t_j)$ for some positive definite $\kappa : T \to R^1$

Definition V.1.1 A s.p. with $T \subset R^1$ and the property that $(X_{t_1+h}, \ldots, X_{t_n+h}) \sim (X_{t_1}, \ldots, X_{t_n})$ for all $\{t_1, \ldots., t_n\} \subset T$ and h such that $\{t_1 + h, \ldots., t_n + h\} \subset T$ is said to be a *strictly stationary process*. ■

- so a weakly stationary Gaussian process is always strictly stationary since $\sigma(t_i + h, t_j + h) = \kappa(t_i - t_j) = \sigma(t_i, t_j)$

Example V.1.1 *Autoregressive process of order 1*

- $\{Z_n : n \in \mathbb{Z}\}$ i.i.d. $N(0, \tau^2)$ and consider

$$X_n = \alpha X_{n-1} + Z_n \tag{1}$$

where X_{n-1} is independent of Z_n

- does there exist a stationary Gaussian process satisfying this?

- assume there is, then

$$\begin{aligned} X_n &= \alpha X_{n-1} + Z_n = \alpha^2 X_{n-2} + \alpha Z_{n-1} + Z_n \\ &\overset{k \text{ steps}}{=} \alpha^k X_{n-k} + \alpha^{k-1} Z_{n-k+1} + \cdots + Z_n \end{aligned}$$

- consider the case when $|\alpha| < 1$ then, since $\{X_n : n \in \mathbb{Z}\}$ is stationary (which implies mean and variance constant), as $k \to \infty$

$$\begin{aligned} E(X_n) &= \alpha^k E(X_{n-k}) = \alpha^k E(X_0) \to 0 \text{ so } E(X_n) = 0 \\ Var(\alpha^k X_{n-k}) &= \alpha^{2k} E(X_{n-k}^2) = \alpha^{2k} E(X_0^2) \to 0 \end{aligned}$$

- therefore, as $k \to \infty$,

$$E\left(\left(X_n - \sum_{j=0}^{k-1} \alpha^j Z_{n-j}\right)^2\right) = \alpha^{2k} E(X_{n-k}^2) \to 0$$

so $X_n - \sum_{j=0}^{k-1} \alpha^j Z_{n-j} \xrightarrow{2} 0$ and it would be natural to define

$$X_n = \sum_{i=0}^{\infty} \alpha^i Z_{n-i}$$

and note that formally such an X_n satisfies (1)

- but is $\sum_{i=0}^{\infty} \alpha^i Z_{n-i}$ a r.v.?
- consider $\sum_{i=0}^{\infty} |\alpha^i Z_{n-i}|$ and let

$$\begin{aligned} A_b &= \left\{\omega : \sum_{i=0}^{\infty} |\alpha^i Z_{n-i}(\omega)| \le b\right\} = \cap_{m=0}^{\infty} \left\{\omega : \sum_{i=0}^{m} |\alpha^i Z_{n-i}(\omega)| \le b\right\} \\ A &= \left\{\omega : \sum_{i=0}^{\infty} |\alpha^i Z_{n-i}(\omega)| = \infty\right\} = \cap_{b=1}^{\infty} \left\{\omega : \sum_{i=0}^{\infty} |\alpha^i Z_{n-i}(\omega)| > b\right\} \end{aligned}$$

- then $A_b, A \in \mathcal{A}$ so $\sum_{i=0}^{\infty} |\alpha^i Z_{n-i}|$ is an (extended) r.v. and by MCT

$$\begin{aligned} E\left(\sum_{i=0}^{n} |\alpha^i Z_{n-i}|\right) &= E(|Z_0|)\sum_{i=0}^{n}|\alpha|^i \uparrow E\left(\sum_{i=0}^{\infty}|\alpha|^i|Z_{n-i}|\right) \\ &= E(|Z_0|)(1-|\alpha|)^{-1} < \infty \end{aligned}$$

so $P(A) = 0$ (otherwise the expectation would be infinite), A can be removed from Ω implying $\sum_{i=0}^{\infty} |\alpha^i Z_{n-i}|$ is a r.v.

- now $X_n^2 = \left(\sum_{i=0}^{\infty} \alpha^i Z_{n-i}\right)^2 = \sum_{i=0}^{\infty} \alpha^{2i} Z_{n-i}^2 + \sum_{i\neq j}^{\infty} \alpha^{i+j} Z_{n-i} Z_{n-j}$

$$\left|\sum_{i\neq j}^{\infty} \alpha^{i+j} Z_{n-i} Z_{n-j}\right| \le \sum_{i\neq j}^{\infty} |\alpha|^{i+j} |Z_{n-i}||Z_{n-j}| \le \sum_{i\neq j}^{\infty} |\alpha|^{i+j} \max\{|Z_{n-i}|^2, |Z$$

$$\le \sum_{i\neq j}^{\infty} |\alpha|^{i+j}\{|Z_{n-i}|^2 + |Z_{n-j}|^2\} \le 2\sum_{i=0}^{\infty} |\alpha|^{2i} Z_{n-i}^2$$

and by MCT

$$E\left(\sum_{i=0}^{k} \alpha^{2i} Z_{n-i}^2\right) = \tau^2 \sum_{i=0}^{k} \alpha^{2i} \uparrow \frac{\tau^2}{1-\alpha^2} < \infty$$

- since $E(X_n^2) < \infty$ for every n, the autocovariance function of $\{X_n : n \in \mathbb{Z}\}$ is defined and is given by (suppose wlog $s \geq t$ and use $E(Z_i Z_j) = 0$ when $i \neq j, E(Z_i^2) = \tau^2$)

$$\begin{aligned}
\sigma(s,t) &= Cov(X_s, X_t) = E\left(\sum_{i=0}^{\infty} \alpha^i Z_{s-i} \sum_{j=0}^{\infty} \alpha^j Z_{t-j}\right) \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \alpha^{i+j} E(Z_{s-i}Z_{t-j}) = \sum_{\{(i,j):\, s-i=t-j\}}^{\infty} \sum^{\infty} \alpha^{i+j} E(Z_{s-i}Z_{t-j}) \\
&= \sum_{i=s-t}^{\infty} \alpha^{2i+t-s} E(Z_{s-i}^2) = \tau^2 \alpha^{s-t} \sum_{i=0}^{\infty} \alpha^{2i} = \frac{\tau^2 \alpha^{|s-t|}}{1-\alpha^2}
\end{aligned}$$

since $j = t - s + j$ so $i + j = t - s + 2i$ and $i = s - t + j \geq s - t$, and it is a weakly stationary process

- for $n_1 < \cdots < n_k \in \mathbb{Z}$ and $\mathbf{a} = (a_1, \ldots, a_k)' \in R^k$ for $Y = \sum_{j=1}^k a_j X_{n_j}$

$$\begin{aligned} c_Y(t) &= E(\exp\{itY\}) \overset{DCT}{=} \lim_{n\to\infty} E\left(\exp\left\{it\sum_{j=1}^k a_j \sum_{m=0}^n \alpha^m Z_{n_j-m}\right\}\right) \\ &= \exp(-\mathbf{a}'(\sigma(n_i, n_j))\mathbf{a}/2) \end{aligned}$$

and so (by Uniqueness) $Y \sim N(0, \mathbf{a}'(\sigma(n_i, n_j)\mathbf{a})$ and Prop. III.9.8 implies that $\{X_n : n \in \mathbb{Z}\}$ is a stationary Gaussian process

- to simulate (approximately) choose $n_0 \in \mathbb{Z}$, say $n_0 = 0$, and choose k s.t.

$$\begin{aligned} \sum_{j=0}^k \alpha^j Z_{n_0-j} &\sim N\left(0, \tau^2 \sum_{j=0}^k \alpha^{2j}\right) = N\left(0, \tau^2 \frac{1-\alpha^{2(k+1)}}{1-\alpha^2}\right) \\ &\approx N\left(0, \frac{\tau^2}{1-\alpha^2}\right) \text{ so take } X_{n_0} = \sum_{j=0}^k \alpha^j Z_{n_0-j} \end{aligned}$$

and then generate $Z_{n_0-k}, Z_{n_0-k+1}, \ldots, Z_{n_0+n} \overset{i.i.d.}{\sim} N(0, \sigma^2)$ and use (1) to obtain $X_{n_0}, X_{n_0+1}, \ldots, X_{n_0+n}$ ■