Exercises Lecture IV

IV.1 Suppose that $\underline{x}$ is a cty point of $F_{\underline{X}}$. This implies that $\lim_{\underline{\epsilon} \downarrow 0} F_{\underline{X}}(\underline{x}+\underline{\epsilon}) = F_{\underline{X}}(\underline{x})$.

This in turn implies that there is no probability mass on the boundary of the set $(-\infty, x_1] \times \cdots \times (-\infty, x_k]$ since $F_{\underline{X}}(\underline{x}) = P_{\underline{X}}((-\infty, x_1] \times \cdots \times (-\infty, x_k])$ and there is no jump. Then by the general of convergence in distribution this implies that $\lim_{n\to\infty} F_{\underline{X}_n}(\underline{x}) = F_{\underline{X}}(\underline{x})$.

IV.2 Note that projection on the i-th coordinate $L_i : \mathbb{R}^k \to \mathbb{R}$ given by $L_i(\underline{x}) = x_i$ is a continuous function. Therefore by the Continuous Mapping Thm for convergence in distribution we have $X_{in} = L_i(\underline{X}_n) \xrightarrow{d} L_i(\underline{X}) = X_i$.

VI.3

We have that

$$E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \mu_1 \;,\; E\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) = \mu_2$$

$$Var(X_1) = \mu_2 - \mu_1 \;,\; Var(X_1^2) = E(X_1^4) - \mu_2^2 = \mu_4 - \mu_2^2$$

$$Cov(X_1, X_1^2) = E(X_1^3) - E(X_1)E(X_1^2) = \mu_3 - \mu_1\mu_2 .$$

Therefore by the CLT

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\begin{pmatrix} X_i \\ X_i^2 \end{pmatrix} - \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}\right) \xrightarrow{d} N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \mu_2-\mu_1^2 & \mu_3-\mu_1\mu_2 \\ \mu_3-\mu_1\mu_2 & \mu_4-\mu_2^2 \end{pmatrix}\right).$$

Put $g(x,y) = y - x^2$ so $G(x,y) = (-2x \;\; 1)$ and

$$g\left(\frac{1}{n}\sum_{i=1}^{n} X_i, \frac{1}{n}\sum_{i=1}^{n} X_i^2\right) = S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 .$$

Therefore, since $g(\mu_1,\mu_2) = \mu_2 - \mu_1^2$,

$$\sqrt{n}\left(S^2 - (\mu_2 - \mu_1^2)\right) \xrightarrow{d} N\left(0, G(\mu_1,\mu_2)(\quad)G^t(\mu_1,\mu_2)\right)$$

and $G(\mu_1,\mu_2)(\quad)G^t(\mu_1,\mu_2)$

$$= \left(-2\mu_1(\mu_2-\mu_1^2) + \mu_3 - \mu_1\mu_2 \quad -2\mu_1(\mu_3-\mu_1\mu_2) + \mu_4 - \mu_2^2\right)(\quad)$$

$$= \left(-3\mu_1\mu_2 + 2\mu_1^3 + \mu_3 \quad -2\mu_1\mu_3 + 2\mu_1^2\mu_2 + \mu_4 - \mu_2^2\right)(\quad)$$

$$= 6\mu_1^2\mu_2 - 4\mu_1^4 - 2\mu_1\mu_3 - 2\mu_1\mu_3 + 2\mu_1^2\mu_2 + \mu_4 - \mu_2^2$$

$$= \mu_4 - 4\mu_1\mu_3 + 8\mu_1^2\mu_2 - 4\mu_1^4 - \mu_2^2$$

$= \mu_4 - 4\mu_1\mu_3 + 8\mu_1^2\mu_2 - 4\mu_1^4 - (\mu_2-\mu_1^2)^2 - 2\mu_1^2\mu_2 + \mu_1^4$

$= \mu_4 - 4\mu_1\mu_3 + 6\mu_1^2\mu_2 - 3\mu_1^4 - (\mu_2-\mu_1^2)^2$

$= E\left(\sum_{k=0}^{4} \binom{4}{k} X^k (-\mu_1)^{4-k}\right) - (\mu_2-\mu_1^2)^2$

$= E\left((X-\mu_1)^4\right) - (\mu_2-\mu_1^2)^2$

and note $E\left((X-\mu_1)^2\right)$ = 4th central moment of X

VI.4 Proof of Prop. VI.10

Since g is continuous this implies that for any sequence $\underset{\sim}{x}_n \to \underset{\sim}{x}$ (pointwise convergence) we have $g(\underset{\sim}{x}_n) \to g(\underset{\sim}{x})$. Now $\underset{\sim}{X}_n \xrightarrow{wp1} \underset{\sim}{X}$ so there is a set $\Omega_0 \subseteq \Omega$ s.t. $\lim_{n\to\infty} \underset{\sim}{X}_n(\omega) = X(\omega)$ for every $\omega \in \Omega_0$ and $P(\Omega_0) = 1$.
So for every $\omega \in \Omega_0$ we have $g(\underset{\sim}{X}_n(\omega)) \to g(\underset{\sim}{X}(\omega))$ which implies $g(\underset{\sim}{X}_n) \xrightarrow{wp1} g(\underset{\sim}{X})$.

VI.5 Proof of Prop. VI.11

This follows for convergence wp1 because $\underset{\sim}{x}_n \to \underset{\sim}{x}$ for pointwise convergence iff $x_{in} \to x_i$ $\forall i = 1, \ldots, k$.

For convergence in mean of order r we have that

$$E(|X_{in} - X_i|^r) \leq E\left(\sum_{i=1}^{k} |X_{in} - X_i|^r\right)$$

$$= \sum_{i=1}^{k} E(|X_{in} - X_i|^r)$$

and so $\underset{\sim}{X}_n \xrightarrow{r} \underset{\sim}{X}$ implies $X_{in} \xrightarrow{r} X_i$ $\forall i = 1, \ldots, k$ and $X_{in} \xrightarrow{r} X_i$ $\forall i = 1, \ldots, k$ implies $E(|X_{in} - X_i|^r) \to 0$ $\forall i = 1, \ldots, k$ which implies $E\left(\sum_{i=1}^{k} |X_{in} - X_i|^r\right) \to 0$ and so $\underset{\sim}{X}_n \xrightarrow{r} \underset{\sim}{X}$.