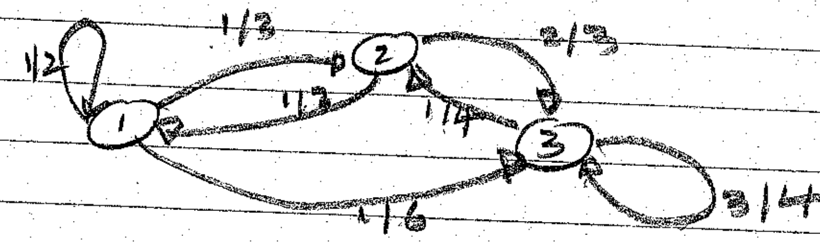
ercises - Lecture 3

III.1.1 Text 1.3.3

(a)



(b) $P(X_0=1) = v_1 = 1/3$

(c) $P(X_0=1, X_1=1) = P(X_0=1)\,P(X_1=1 \mid X_0=1)$
$= (1/3)(1/2) = 1/6$

(d) $P(X_0=1, X_1=2) = (1/3)(1/3) = 1/9$

(e) $P(X_0=1, X_1=1, X_2=1) = (1/3)(1/2)(1/2) = 1/12$

(f) $P(X_0=1, X_1=1, X_2=1, X_3=2) = (1/3)(1/2)(1/2)(1/6)$
$= 1/72$

III.1.2 Text 1.3.9

For Ehrenfest urn with $d=5$

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1/5 & 0 & 4/5 & 0 & 0 & 0 \\ 0 & 2/5 & 0 & 3/5 & 0 & 0 \\ 0 & 0 & 3/5 & 0 & 2/5 & 0 \\ 0 & 0 & 0 & 4/5 & 0 & 1/5 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

(a) $P(X_0=2, X_1=2) = (1)(0) = 0$

(b) $P(X_0=2, X_1=1) = (1)(2/5) = 2/5$

(c) $P(X_0=2, X_1=1, X_2=0)$
$= P(X_0=2)\, P(X_1=1 \mid X_0=2)\, P(X_2=0 \mid X_1=1)$
$= (1)(2/5)(1/5)(1) = 2/10$

(d) $P(X_0=2, X_1=1, X_2=0, X_3=1)$
$= (1)(2/5)(1/5)(1) = 2/10$

III.1.3 Text 1.4.7

The paths from $1 \to 2$ in 3 steps

$1 \to 0 \to 1 \to 2$ with prob. $(1/5)(1)(4/5) = 4/25$

$1 \to 2 \to 1 \to 2$ " " $(4/5)(2/5)(4/5) = 32/125$

$1 \to 2 \to 3 \to 2$ " " $(4/5)(3/5)(3/5) = 36/125$

$\therefore P_{12}^{(3)} = 4/25 + 32/125 + 36/125 = 88/125$

III.1.4 Proof of Chapman-Kolmogorov (i)

$$P_{ij}^{(m+n)} = P(X_{m+n}=j \mid X_0=i)$$

$$= \sum_{k\in S} P(X_{m+n}=j, X_m=k \mid X_0=i)$$

$$= \sum_{k\in S} P(X_{m+n}=j \mid X_0=i, X_m=k)\, P(X_m=k \mid X_0=i)$$

$$\overset{\text{MP, TH}}{=} \sum_{k\in S} P(X_n=j \mid X_0=k)\, P(X_m=k \mid X_0=i)$$

$$= \sum_{k\in S} P_{ik}^{(m)}\, P_{kj}^{(n)}$$

M. 2.2 Text 1.5.12

(a) paths from 4 to 3 in 3 steps

$4 \to 1 \to 1 \to 3$ $\quad (1/2)(1/3)(1/2) = 1/12$
$4 \to 1 \to 2 \to 3$ $\quad (1/2)(1/6)(3/4) = 1/16$
$4 \to 2 \to 1 \to 3$ $\quad (1/2)(1/4)(1/2) = 1/16$

$$\therefore \; P_{43}^{(3)} = 1/12 + 1/16 + 1/16 = 10/48$$

(b) $f_{23} = P_{23} + P_{21} f_{13} + P_{22} f_{23} + P_{24} f_{43}$

using 1.5.10 in text $\quad = 3/4 + (1/4) f_{13}$ ①

$f_{13} = P_{13} + P_{11} f_{13} + P_{12} f_{23} + P_{14} f_{43}$

$= 1/2 + (1/3) f_{13} + (1/6) f_{23}$ ②

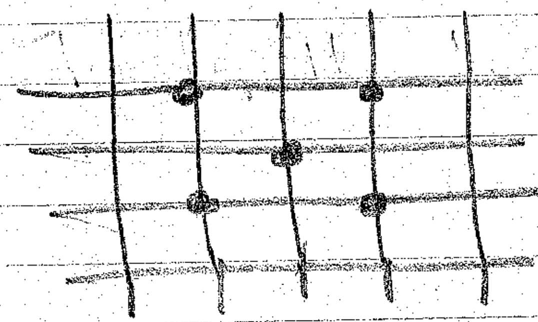
$\therefore$ from ② $(2/3) f_{13} = 1/2 + (1/6) f_{23}$ or
$f_{13} = (3/4) + (1/4) f_{23}$

$\therefore$ from ① $f_{23} = (3/4) + (1/4)\left((3/4) + (1/4) f_{23}\right)$
$= (3/4 + 3/16) + (1/16) f_{23}$

$$\therefore \; f_{23} = \frac{16}{15}\left(\frac{3}{4} + \frac{3}{16}\right) = \frac{16}{15}\,\frac{15}{16} = 1$$

(c) $f_{21} = P_{21} + P_{22} f_{21} + P_{23} f_{31} + P_{24} f_{41}$
$= (1/4) + (0) f_{21} + (3/4) f_{31} + 0\, f_{41}$
$= (1/4) + 3/4\, f_{31}$
$f_{31} = P_{31} + P_{32} f_{21} = \frac{2}{5} + \frac{3}{5} f_{21}$ and solving this gives $f_{21} = 1$

M.2.3 Text 1.5.15



Start the chain at $(0,\dots,0)$ and note $S + \mathbb{Z}^d$

$$P(X_n = (0,0,\dots,0) \mid X_0 = (0,\dots,0))$$

$$= \frac{P(X_{01}=0,\dots,X_{0d}=0,\ X_{n1}=0,\dots,X_{nd}=0)}{P(X_{01}=0,\dots,X_{0d}=0)}$$

ind.

$$= \frac{P(X_{01}=0, X_{n1}=0)\cdots P(X_{0d}=0, X_{nd}=0)}{P(X_{01}=0)\cdots P(X_{0d}=0)}$$

$$= P(X_{n1}=0 \mid X_{01}=0)\cdots P(X_{nd}=0 \mid X_{0d}=0)$$

$$= (P_{00}^{(n)})^d$$

Then $\displaystyle\sum_{n=1}^{\infty} (P_{00}^{(n)})^d = \sum_{n=1}^{\infty} \left(\binom{2n}{n}\left(\tfrac{1}{2}\right)^{2n}\right)^d$

and using the notation of Lecture 26 p.7,8

$$= \sum_{n=1}^{\infty} \left(\binom{2n}{n}\frac{1}{b_n}\right)^d \left(\frac{1}{\sqrt{\pi n}}\right)^d$$

and since $\binom{2n}{n}\frac{1}{b_n} \to 1$ as $n \to \infty$ this series diverges or converges like

$\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}}\right)^d$. For $d=1$ this series is

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$ and when $d=2$ the series is $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ while for $d>2$ the series is $\sum_{n=1}^{\infty} \frac{1}{n^{d/2}} \le \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ and $\int_1^{\infty} x^{-3/2}\,dx = -2x^{-1/2}\Big|_1^{\infty} = \frac{1}{2} < \infty$ and so by the integral test the series converges. This implies the chain is recurrent when $d=1,2$ but transient when $d>2$.

__Note__ This chain is irreducible on a subset of $\mathbb{Z}^d$. A more common definition is to make the transitions $X_n + Z_{n+1}$ where Z_i is uniformly distributed on $\{\underset{\sim}{e}_1, -\underset{\sim}{e}_1, \underset{\sim}{e}_2, -\underset{\sim}{e}_2, \ldots, \underset{\sim}{e}_d, -\underset{\sim}{e}_d\}$ where $\underset{\sim}{e}_i$ = i-th standard basis vector. This chain is irreducible on $\mathbb{Z}^d$ and the same result holds.

III.3.1 Text 1.6.4

Frog walk: Suppose $i < j$, then $p_{i,i+1} p_{i+1,i+2} \cdots p_{j-1,j}$
$= (1/3)^{j-i} > 0$ and $p_{20,1} = 1/3 > 0$
and similarly if $i > j$ so chain is irreducible.

Simple random walk: If $i < j$ then $p_{i,i+1} \cdots p_{j-1,j}$
$= p^{j-i} > 0$ and similarly if $j < i$ so the chain is irreducible.

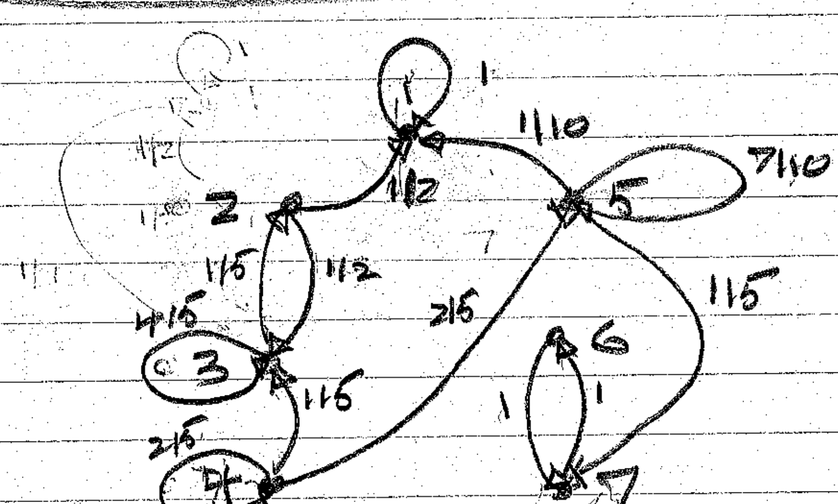
Ehrenfest's urn: If $i < j$ then $p_{i,i+1} \cdots p_{j-1,j}$
$= \frac{d-i}{d} \frac{d-i+1}{d} \cdots \frac{d-j+1}{d} > 0$
and so the chain is irreducible.

III.3.2 Text 1.6.20

see solution in text

III.3.3 Text 1.6.24

(a)



(b) The sets $\{1\}$, $\{6,7\}$ are closed and these states are recurrent and all the other states are transient as there is a positive probability of entering one of those sets from 2, 3, 4 and 5

(c) $f_{11} = 1$ otherwise use (1.5.10 in text)

$$f_{21} = p_{21} + p_{22}f_{21} + p_{23}f_{31} + p_{24}f_{41} + p_{25}f_{51} + p_{26}f_{61} + p_{27}f_{71}$$

$$= \tfrac{1}{2} + 0 + \tfrac{1}{2}f_{31} + 0 + 0 + 0 + 0$$

$$= \tfrac{1}{2} + \tfrac{1}{2}f_{31}$$

$$f_{31} = p_{31} + p_{32}f_{21} + p_{33}f_{31} + p_{34}f_{41} + p_{35}f_{51} + p_{36}f_{61} + p_{37}f_{71}$$

$$= 0 + (\tfrac{1}{5})f_{21} + (\tfrac{4}{5})f_{31} = \tfrac{1}{5}f_{21} + \tfrac{4}{5}f_{31}$$

$$\therefore \tfrac{1}{5}f_{31} = \tfrac{1}{5}f_{21} \text{ or } f_{31} = f_{21} \text{ and so}$$

$$f_{21} = \tfrac{1}{2} + \tfrac{1}{2}f_{21} \text{ so } f_{21} = 1 = f_{31}$$

$$f_{51} = p_{51} + p_{52}f_{21} + p_{53}f_{31} + p_{54}f_{41} + p_{55}f_{51} + p_{56}f_{61} + p_{57}f_{71}$$

$$f_{51} = \tfrac{1}{10} + 0 + 0 + 0 + \tfrac{7}{10}f_{51} + 0 + \tfrac{1}{5}0 = \tfrac{1}{10} + \tfrac{7}{10}f_{51}$$

$$\therefore \tfrac{3}{10}f_{51} = \tfrac{1}{10} \text{ or } f_{51} = \tfrac{1}{3}$$

$f_{61} = f_{71} = 0$ because $\{6,7\}$ is closed

$$f_{41} = p_{41} + p_{42}f_{21} + p_{43}f_{31} + p_{44}f_{41} + p_{45}f_{51} + 0 + 0$$

$$= 0 + 0 + \left(\tfrac{1}{5}\right)(1) + \tfrac{3}{5}f_{41} + \tfrac{2}{5}\cdot\tfrac{1}{3}$$

$$\therefore \tfrac{2}{5}f_{41} = \tfrac{1}{5} + \tfrac{2}{15} \text{ and } f_{41} = \tfrac{1}{2} + \tfrac{1}{3} = \tfrac{5}{6}$$

VI.4.1 Text 1.7.2

$$\lim_{p\to 1/2} \frac{\left(\frac{1-p}{p}\right)^a - 1}{\left(\frac{1-p}{p}\right)^c - 1} = \lim_{r\to 1} \frac{r^a - 1}{r^c - 1}$$

L'Hopital

$$= \lim_{r\to 1} \frac{a r^{a-1}}{c r^{c-1}} = \frac{a}{c} \lim_{r\to 1} r^{-(c-a)} = \frac{a}{c}$$

so $s(a)$ is continuous.

VI.4.2 Text 1.7.4

If $p = 1/2$ $\quad r(a) + s(a) = \frac{c-a}{c} + \frac{a}{c} = 1$ and if $p \neq \frac{1}{2}$

$$r(a) + s(a) = \frac{\left(\frac{p}{1-p}\right)^{c-a} - 1}{\left(\frac{p}{1-p}\right)^c - 1} + \frac{\left(\frac{1-p}{p}\right)^a - 1}{\left(\frac{1-p}{p}\right)^c - 1}$$

take common denominator

$$= \Big\{ \cancel{\left(\frac{p}{1-p}\right)^{c-a}\left(\frac{1-p}{p}\right)^c} - \left(\frac{1-p}{p}\right)^c - \cancel{\left(\frac{p}{1-p}\right)^{c-a}} + 1 + \cancel{\left(\frac{1-p}{p}\right)^a\left(\frac{p}{1-p}\right)^c}$$

$$- \left(\frac{p}{1-p}\right)^c - \cancel{\left(\frac{1-p}{p}\right)^a} + 1 \Big\} \Big/ \Big\{ 2 - \left(\frac{p}{1-p}\right)^c - \left(\frac{1-p}{p}\right)^c \Big\}$$

$$= 1$$

11.4.3 Text 1.7.9

(a) We need to compute $s(7) = P_7(T_{10} < T_0)$

$= P_a(X_n = 10 \text{ for some } n\ X_1, \dots, X_{n-1} \neq 0)$

$$= \frac{(3/5 / 2/5)^7 - 1}{(3/5/2/5)^{10} - 1} = \frac{(\frac{3}{2})^7 - 1}{(3/2)^{10} - 1} = 0.284$$

(b) $s(7) = P_7(T_{20} < T_0)$

$$= \frac{(3/2)^7 - 1}{(3/2)^{20} - 1} = 0.005$$

(c) $P_7(T_0 < \infty) = 1$ so $P_7(T_0 = \infty) = 0$

11.4.4 Text 1.7.11 With $r = \frac{1-p}{p} \geq 0$ then

$$S_{10,000}(9700) = \frac{r^{9700} - 1}{r^{10000} - 1} \geq 1/2 \text{ iff}$$

$r^{9700} \geq \frac{1}{2} r^{10000} + \frac{1}{2}$ iff $r^{9700}\left(1 - \frac{1}{2} r^{300}\right) \geq 1/2$

which gives equality when $r = 1$. Now

$$\frac{d\, r^a(1 - \frac{1}{2} r^b)}{dr} = r^{a-1}\left(a - \frac{(a+b)}{2} r^b\right) = 0$$

when $r = \left(\frac{2a}{a+b}\right)^{1/b} \overset{\substack{a=9700 \\ b=300}}{=} 1.00224$ and $r = 0$ and so using the fact that this derivative equals $\frac{a-b}{2} = 4700 > 0$ when $r = 1$, we have that $r^{9700}(1 - \frac{1}{2}r^{300})$ increases for $0 < r < 1.002211$ and decreases for $r > 1.002211$ and so achieves a local max at $r = 1.002211$. Since $r^{9700}(1 - \frac{1}{2}r^{300}) = \frac{1}{2}$ when $r = 1$ this implies that $r^{9700}(1 - \frac{1}{2}r^{300}) > \frac{1}{2}$ when $r = 1.002211$.

Therefore, since $p = \frac{1}{1+r}$ is decreasing in r this implies the smallest value of p satisfying the inequality occurs for the largest value of r in $(1.002211, \infty)$ satisfying the inequality. I solved for this by bisection which gives the approximate value $r = 1.002313$ which corresponds to probability $p = \frac{1}{1+r} = 0.4994224$, and this is the smallest probability satisfying the inequality. Note that this is $\approx 1/2$.

So a gambler betting with slightly less than even odds and with capital 9700 can bankrupt a house with capital 10000 with probability $\frac{1}{2}$.