Exercise Solutions Lecture 5a-5d

V.1 $E(Y h(\underset{\sim}{x})) = \int_{\mathbb{R}^1}\int_{\mathbb{R}^k} y\, h(\underset{\sim}{x})\, f_{(\underset{\sim}{x},Y)}(\underset{\sim}{x},y)\, dy\, d\underset{\sim}{x}$

$= \int_{\mathbb{R}^k}\int_{\mathbb{R}^1} y\, h(\underset{\sim}{x}) \dfrac{f_{(\underset{\sim}{x},Y)}(\underset{\sim}{x},y)}{f_{\underset{\sim}{x}}(\underset{\sim}{x})}\, f_{\underset{\sim}{x}}(\underset{\sim}{x})\, dy\, d\underset{\sim}{x}$

since $E|Y h(\underset{\sim}{X})| < \infty$ we can change order of integration

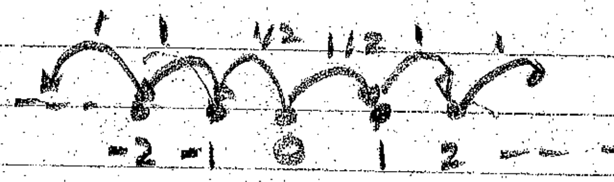
$= \int_{\mathbb{R}^k} \left(\int_{\mathbb{R}^1} y\, f_Y(y|\underset{\sim}{x})\, dy \right) h(\underset{\sim}{x})\, f_{\underset{\sim}{x}}(\underset{\sim}{x})\, d\underset{\sim}{x}$

$= \int_{\mathbb{R}^k} E(Y|\underset{\sim}{X})(\underset{\sim}{x})\, h(\underset{\sim}{x})\, f_{\underset{\sim}{x}}(\underset{\sim}{x})\, d\underset{\sim}{x}$

$= E\left(h(\underset{\sim}{X})\, E(Y|\underset{\sim}{X}) \right)$

V.2
$$P = \begin{array}{c|c} & \cdots \; -1 \;\; 0 \;\; 1 \; \cdots \\ \hline \begin{array}{c} \vdots \\ -1 \\ 0 \\ 1 \\ \vdots \end{array} & \begin{pmatrix} \ddots & & & & & \\ & 1 & 0 & 0 & \cdots & \\ & 1/2 & 0 & 1/2 & & \\ & 0 & 0 & 0 & 1 & 0 \; \cdots \\ & & & & & \ddots \end{pmatrix} \end{array}$$

(a)



(b) $P(X_n = 0) = P(X_n = 0 | X_0 = 0) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$

$P(X_n = 1) = P(X_n = 1 | X_0 = 0) = \begin{cases} 1/2 & n = 1 \\ 0 & \text{otherwise} \end{cases}$

For $i > 1$ $P(X_n = i) = P(X_n = i | X_0 = 0) = \begin{cases} 1/2 & n = i \\ 0 & \text{otherwise} \end{cases}$

$P(X_n = -i) = P(X_n = -i | X_0 = 0) = \begin{cases} 1/2 & n = i \\ 0 & \text{otherwise} \end{cases}$

For $i \leq -1$ $P(X_n = i) = P(X_n = i \mid X_0 = 0) = \begin{cases} 1/2 & n = -i \\ 0 & \text{otherwise} \end{cases}$

(c) $E(X_n) = 0$ when $n = 0$

$= \frac{1}{2}(-1) + \frac{1}{2}(1) = 0$ when $n = 1$

$= \frac{1}{2}(-n) + \frac{1}{2}(n) = 0$ when $n > 1$

(d) From (c) $E(X_n) = E(X_0) = 0$

(e) $\sum_j j P_{1j} = \sum_{j \neq 2} j \cdot 0 + 2 P_{12} = 2$

(f) $P(X_2 = i \mid X_0 = 0, X_1 = 1) = \begin{cases} 0 & i \neq 2 \\ 1 & i = 2 \end{cases}$

$E(X_2 \mid X_0 = 0, X_1 = 1) = 2$

(g) $E(X_2 \mid X_0 = 0, X_1 = 1) \neq X_1$ which is 1

so the process is not a martingale.

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V.1 $E(Yh(\underset{\sim}{x})) = \int_{\mathbb{R}^1}\int_{\mathbb{R}^k} y\,h(\underset{\sim}{x})\,f_{(\underset{\sim}{x},Y)}(\underset{\sim}{x},y)\,dy\,d\underset{\sim}{x}$

$= \int_{\mathbb{R}^k}\int_{\mathbb{R}^1} y\,h(\underset{\sim}{x})\,\dfrac{f_{(\underset{\sim}{x},Y)}(\underset{\sim}{x},y)}{f_{\underset{\sim}{x}}(\underset{\sim}{x})}\,f_{\underset{\sim}{x}}(\underset{\sim}{x})\,dy\,d\underset{\sim}{x}$

since $E|Yh(\underset{\sim}{x})| < \infty$ we can change order of integration

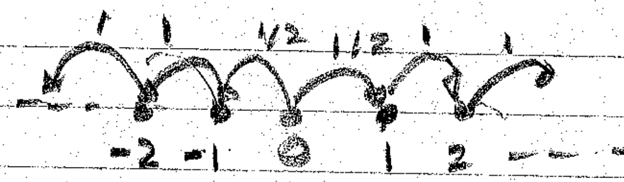
$= \int_{\mathbb{R}^k}\left(\int_{\mathbb{R}^1} y\,f_Y(y|\underset{\sim}{x})\,dy\right) h(\underset{\sim}{x})\,f_{\underset{\sim}{x}}(\underset{\sim}{x})\,d\underset{\sim}{x}$

$= \int_{\mathbb{R}^k} E(Y|\underset{\sim}{x})(\underset{\sim}{x})\,h(\underset{\sim}{x})\,f_{\underset{\sim}{x}}(\underset{\sim}{x})\,d\underset{\sim}{x}$

$= E\left(h(\underset{\sim}{x})\,E(Y|\underset{\sim}{x})\right)$

V.2 $P = \begin{array}{c|c} & \cdots\ -1\ \ 0\ \ 1\ \cdots \\ \hline \vdots & \\ -1 & \\ 0 & \\ 1 & \\ \vdots & \end{array} \begin{pmatrix} \ddots & & & & & \\ & 1 & & & & \\ & 1 & 0 & 0 & \cdots & \\ & 1/2 & 0 & 1/2 & & \\ & 0 & 0 & 0 & 1 & 0 \cdots \\ & & & & & \ddots \end{pmatrix}$

(a)



(b) $P(X_n=0) = P(X_n=0|X_0=0) = \begin{cases} 1 & n=0 \\ 0 & \text{otherwise} \end{cases}$

$P(X_n=1) = P(X_n=1|X_0=0) = \begin{cases} 1/2 & n=1 \\ 0 & \text{otherwise} \end{cases}$

for $i\geq 1$ $P(X_n=i) = P(X_n=i|X_0=0) = \begin{cases} 1/2 & n=i-1 \\ 0 & \text{otherwise} \end{cases}$

$P(X_n=-1) = P(X_n=-1|X_0=0) = \begin{cases} 1/2 & n=1 \\ 0 & \text{otherwise} \end{cases}$

For $i \leq -1$ $P(X_n = i) = P(X_n = i \mid X_0 = 0) = \begin{cases} 1/2 & n = -i-1 \\ 0 & \text{otherwise} \end{cases}$

(c) $\mathbb{E}(X_n) = 0$ when $n = 0$

$= \frac{1}{2}(-1) + \frac{1}{2}(1) = 0$ when $n = 1$

$= \frac{1}{2}(-n) + \frac{1}{2}(n) = 0$ when $n > 1$

(d) From (c) $\mathbb{E}(X_n) = \mathbb{E}(X_0) = 0$

(e) $\sum_j j\,p_{1j} = \sum_{j \neq 2} j \cdot 0 + 2p_{12} = 2$

(f) $P(X_2 = i \mid X_0 = 0, X_1 = 1) = \begin{cases} 0 & i \neq 2 \\ 1 & i = 2 \end{cases}$

$\mathbb{E}(X_2 \mid X_0 = 0, X_1 = 1) = 2$

(g) $\mathbb{E}(X_2 \mid X_0 = 0, X_1 = 1) \neq X_1$ which is 1

$\therefore$ the process is not a martingale.

V.3 $T = \begin{cases} n & \text{if } X_0 \neq -5, \dots, X_{n-1} \neq -5, X_n = -5 \text{ and } n \leq 10^{12} \\ 10^{12} & \text{if } X_0 \neq -5, \dots, X_{10^{12}} \neq -5 \text{ and } n > 10^{12} \end{cases}$

$\therefore \{T = n\} \in \mathcal{A}_{X_0, \dots, X_n}$ since its truth depends only on $X_0, \dots, X_n$.

V.4 $T_i = n$ if $X_0 \neq i, \dots, X_{n-1} \neq i, X_n = i$

and so $\{T_i = n\} \in \mathcal{A}_{X_0, \dots, X_n}$ since its truth depends only on $X_0, \dots, X_n$.

V.5 $\{\min(T_1, T_2) = n\} = \{T_1 = n, T_2 \geq n\} \cup \{T_2 = n, T_1 > n\}$

$= (\{T_1 = n\} \cap \{T_2 \geq n\}) \cup (\{T_2 = n\} \cap \{T_1 > n\})$

$= (\{T_1 = n\} \cap \{T_2 \leq n\}^c) \cup (\{T_2 = n\} \cap \{T_1 \leq n\}^c)$

$\in \mathcal{A}_{X_0, \dots, X_n}$ since all of these events are in $\mathcal{A}_{X_0, \dots, X_n}$ since T_1, T_2 are stopping times

$\{\max(T_1, T_2) = n\} = \{T_1 = n, T_2 \leq n\} \cup \{T_1 \leq n, T_2 = n\}$
$\in \mathcal{A}_{X_0, \dots, X_n}$ as with previous

$\{T_1 T_2 = n\} = \bigcup_{\substack{\{(a,b)\,:\, ab = n \\ a, b \in \mathbb{N}_0\}}} \{T_1 = a, T_2 = b\} = \bigcup_{\{\ \}} \{T_1 = a\} \cap \{T_2 = b\}$

$\in \mathcal{A}_{X_0, \dots, X_n}$ since $a, b \in \{0, \dots, n\}$

$$\{T_1+T_2=n\} = \bigcup_{\substack{(a,b):a+b=n\\ a,b\in\mathbb{N}_0}} \{T_1=a, T_2=b\} \in \mathcal{A}_{X_0,\dots,X_n}.$$

V.6

(a) $X_n = 5 + Z_1 + \dots + Z_n = X_{n-1} + Z_n$

$\therefore E(X_n \mid X_0,\dots,X_{n-1}) = X_{n-1} + E(Z_n \mid X_0,\dots,X_{n-1})$

$= X_{n-1} + E(Z_n)$ since Z_n is ind. of $X_0,\dots,X_{n-1}$

$= X_{n-1} - \frac{3}{4} + \frac{c}{4} = X_{n-1}$

iff $\frac{c}{4} = \frac{3}{4}$ or $c=3$

(b) $E(X_9) = E(X_0) = 5$ since it is a martingale and $E(X_n) = E(E(X_n \mid X_0,\dots,X_{n-1})) = E(X_{n-1})$ $= \dots = E(X_0) = 5$

(c) $T = \inf\{n\geq 1 : X_n = 0 \text{ or } Z_n > 0\}$

$= \inf\{n\geq 1 : X_n = 0 \text{ or } X_n - X_{n-1} > 0\} \in \mathcal{A}_{X_0,\dots,X_n}$

Suppose $Z_1 = Z_2 = \dots = Z_5 = -1$, then $X_n = 0$, so $T=5$ and if $Z_i = 3$ for one of $i = 1,\dots,5$ then $T = i \leq 5$. Therefore T is bounded. Therefore, by the Optional Stopping Lemma $E(X_T) = E(X_0) = 5$.

IV.7 (a) We have that $|X_n| I_{n \le U} \le 6$ since, if $U < n$ then $|X_n| I_{n \le U} = 0$ and if $U \ge n$ then $X_n \in \{-6, -5, \ldots, 0, 1, 2, 3, 4\}$ otherwise if X_n is not in this set then for some $m < n$ we must have had $X_m = 4$ or $X_m = -6$ which would imply $U \le n$. Therefore by the Optional Stopping Corollary $E(X_U) = E(X_0) = 0$.

(b) Note $P_0(T_4 < \infty) = 1$ since the chain is recurrent and $X_{T_4} = 4$. Therefore, $E(X_{T_4}) = 4 \ne E(X_0) = 0$.