(6)

II.18 (a) Suppose player 1 bets \$1 on 1st toss = H and, if they win bets \$2 on next toss = H and otherwise quits. Player 2 bets \$1 on the 2nd toss = H, etc until HH occurs and the game stops. Let X_n = total winnings at toss n so $\{X_n; n=0,1,2,\ldots\}$ is a martingale which implies $E(X_T) = E(X_0) = 0$. But $X_T = -(T-2)+3+1 = 6-T$. Therefore $E(6-T) = E(X_T) = 0$ so $E(T) = 6$.

(b) $X_T = -(T-3)+7+3+1 = 14-T$ so $E(T)=14$

(c) $X_T = -(T-4)+15-1+\cancel{3}\ \cancel{1} = \overset{20}{\cancel{\ }} - T$ so $E(T) = 20$

(d) $X_T = -(T-4)+15-1-1+1 = 18-T$ so $E(T) = 18$

II.19 $E(Y_n \mid Y_0, \ldots, Y_{n-1})$

$= E\left(E(Y_n \mid X_0, \ldots, X_{n-1}) \mid Y_0, \ldots, Y_{n-1}\right)$

since $\mathcal{F}_{Y_0,\ldots,Y_n} \subseteq \mathcal{F}_{X_0,\ldots,X_n}$

and $E(Y_n \mid X_0, \ldots, X_{n-1}) = E(X_0 - X_n \mid X_0, \ldots, X_{n-1})$

$= X_0 - X_{n-1}\ E(X_n) = X_0 - X_{n-1} = Y_{n-1}$

$= E(Y_{n-1} \mid Y_0, \ldots, Y_{n-1}) = Y_{n-1}$

V.10 By Jensen's inequality

$$\mathbb{E}\left(g(Y_n) \mid Y_0,\dots,Y_{n-1}\right) \geq g\left(\mathbb{E}(Y_n \mid Y_0,\dots,Y_{n-1})\right)$$

$= g(Y_{n-1})$ since $\{Y_n : n \geq 0\}$ is a martingale.

Now since $\mathcal{A}_{g(Y_0),\dots,g(Y_n)} \subseteq \mathcal{A}_{Y_0,\dots,Y_n}$ $\forall n$

we have that (using Prop V.14)

$$\mathbb{E}\left(g(Y_n) \mid g(Y_0),\dots,g(Y_{n-1})\right)$$

$$= \mathbb{E}\left(\mathbb{E}(g(Y_n) \mid Y_0,\dots,Y_{n-1}) \mid g(Y_0),\dots,g(Y_{n-1})\right)$$

$$= \mathbb{E}\left(g(Y_{n-1}) \mid g(Y_0),\dots,g(Y_{n-1})\right) = g(Y_{n-1})$$

V.11 $\mathbb{E}\left((X_{n+1}-X_n)^2 \mid X_0,\dots,X_n\right)$

$$= \mathbb{E}\left(X_{n+1}^2 - 2X_nX_{n+1} + X_n^2 \mid X_0,\dots,X_n\right)$$

$$= \mathbb{E}(X_{n+1}^2 \mid X_0,\dots,X_n) - 2X_n\mathbb{E}(X_{n+1} \mid X_0,\dots,X_n) + X_n^2$$

$$= \mathbb{E}(X_{n+1}^2 \mid X_0,\dots,X_n) - 2X_n^2 + X_n^2$$ since $\{X_n\}$ is a martingale, $\mathbb{E}(X_{n+1} \mid X_0,\dots,X_n) = X_n$

$$= \mathbb{E}(X_{n+1}^2 \mid X_0,\dots,X_n) - X_n^2$$

IV.12 $E((X_n - X_k)X_j)$

$= E(E((X_n - X_k)X_j \mid X_0, \dots, X_k))$

$= E(E(X_n X_j \mid X_0, \dots, X_k)) - E(X_j X_k)$

$= E(X_j E(X_n \mid X_0, \dots, X_k)) - E(X_j X_k)$

$= E(X_j X_k) - E(X_j X_k) = 0$ since

$E(X_n \mid X_0, \dots, X_k)$

$= E(E(X_n \mid X_0, \dots, X_{n-1}) \mid X_0, \dots, X_k)$

$= E(X_{n-1} \mid X_0, \dots, X_k)$ since $\mathcal{A}_{X_0, \dots, X_{n-1}}$

$= \dots = X_k$ $\qquad \supseteq \mathcal{A}_{X_0, \dots, X_k}$

$\therefore E((X_n - X_k)(X_j - X_i))$

$= E((X_n - X_k)X_j) - E((X_n - X_k)X_i) = 0 - 0 = 0$

Now $X_n - X_0 = \sum_{k=1}^{n} (X_k - X_{k-1})$ and

$(X_n - X_0)^2 = \sum_{k=1}^{n} (X_k - X_{k-1})^2 + 2\sum_{i<j} (X_j - X_{j-1})(X_i - X_{i-1})$

Therefore $E(X_n - X_0)^2 = \sum_{k=1}^{n} E(X_k - X_{k-1})^2$ since

by the above $E(X_j - X_{j-1})(X_i - X_{i-1}) = 0$.

IV.13

(a)

X_{n-1}	$E(X_n \mid X_{n-1})$
5	5
6	$1/2 \cdot 5 + \frac{1}{2} 7 = 6$
7	$(2/7)\,5 + (2/7)\,6 + (3/7)\,9 = 7$
$\vdots$	
m	$\dfrac{2(m-2) + 2(m-1) + 3(m+2)}{7}$
	$= \dfrac{7m - 4 - 2 + 6}{7} = m$

$\therefore \{X_n : n \geq 0\}$ is a martingale.

(b) Since $X_n \geq 5$ $\forall n$ the MCT implies $\exists$ r.v. X s.t. $X_n \xrightarrow{wp1} X$. This implies that for every realization of the chain ω $\exists N$ s.t. $\forall n \geq N$ $X_n(\omega) = i$ for some $i \in \{5, 6, 7, \ldots\}$. But if $i \neq 5$ then there is a positive probability that $X_{n+1}(\omega) \neq i$. Therefore $P(X = 5) = 1$.

V.14 (a) $E(Y_n \mid Y_0, \dots, Y_{n-1}) = E(Y_n \mid Y_{n-1})$
since it is Markov

$$= \frac{1}{n+r+b} E(R_n \mid R_{n-1})$$

$$= \frac{1}{n+r+b}\left[(R_{n-1}+1)\frac{R_{n-1}}{n-1+r+b} + R_{n-1}\left(1-\frac{R_{n-1}}{n-1+r+b}\right)\right]$$

$$= \frac{1}{n+r+b}\left[\frac{R_{n-1}}{n-1+r+b} + R_{n-1}\right]$$

$$= \frac{R_{n-1}}{n-1+r+b}\left[\frac{1+n-1+r+b}{n+r+b}\right] = Y_{n-1}$$

(b) $Y_n \geq 0 \ \forall n$ and so $Y_n \xrightarrow{wp1} Y$ for some r.v. Y by the MCT

V.16 $P(Z_i = 0) = \frac{2}{3}$, $P(Z_i = 2) = \frac{1}{3}$

(a) $P(X_1 = 0) = P(Z_{10} + Z_{20} = 0)$ since $X_0 = 2$

$= P(Z_{10} = Z_{20} = 0) = 4/9$

(b) $P(X_1 = 2) = P(Z_{10} + Z_{20} = 2)$

$= P(Z_{10} = 0, Z_{20} = 2) + P(Z_{10} = 1, Z_{20} = 1) + P(Z_{10} = 2, Z_{20} = 0)$

$= \frac{2}{3}\frac{1}{3} + 0 + \frac{1}{3}\cdot\frac{2}{3} = 4/9$

(c) $P(Z_{11} + Z_{21} = 4) = P(Z_{11} = Z_{21} = 2) = \frac{1}{9}$

$$P(X_2=8) = P(X_2=8 \mid X_1=0)P(X_1=0) + P(X_2=8 \mid X_1=2)P(X_1=2) + P(X_2=8 \mid X_1=4)P(X_1=4)$$

$$= 0\cdot\frac{4}{9} + 0\cdot\frac{4}{9} + P(Z_{11}+Z_{21}+Z_{31}+Z_{41}=8)\frac{1}{9}$$

$$= P(Z_{11}=Z_{21}=Z_{31}=Z_{41}=2)\frac{1}{9}$$

$$= \left(\frac{1}{3}\right)^6$$

(d) $\mu = 0\cdot\frac{2}{3} + 2\cdot\frac{1}{3} = \frac{2}{3}$ and P(

$\therefore P(T<\infty) = P(X_n = 0 \text{ for some } n) = 1$

by ~~Case~~ 1 established in class.

V.16 $X_0 = 100$, $K = 110$, $P(X_S = 80) = \frac{9}{10}$, $P(X_S = 130) = \frac{1}{10}$

$\therefore$ by Prop 3.7.7 in the book, the no-arbitrage price of a call at time S for K is

$$\frac{(100-80)(130-110)}{130-80} = \frac{(20)(20)}{50} = 8$$