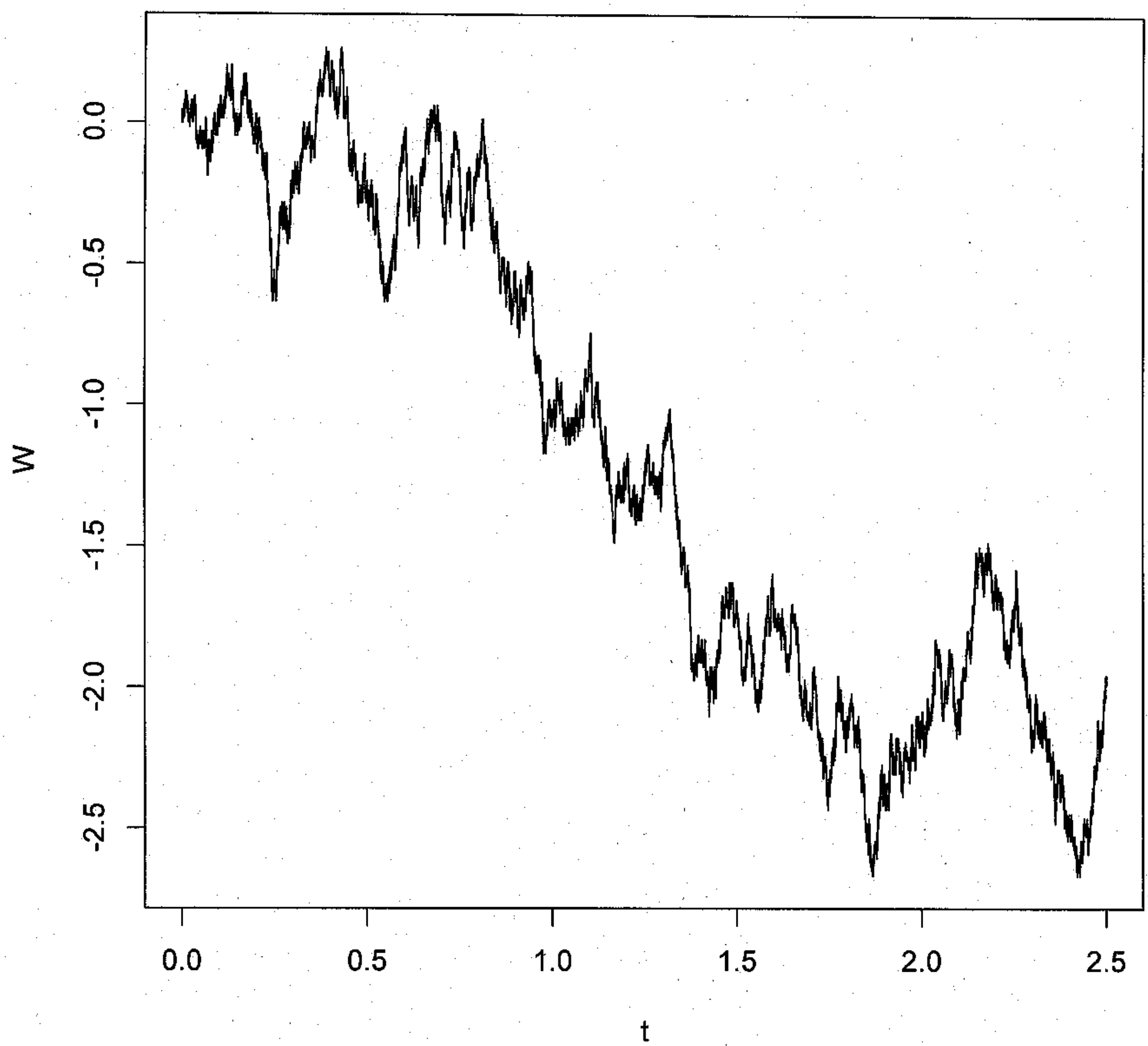
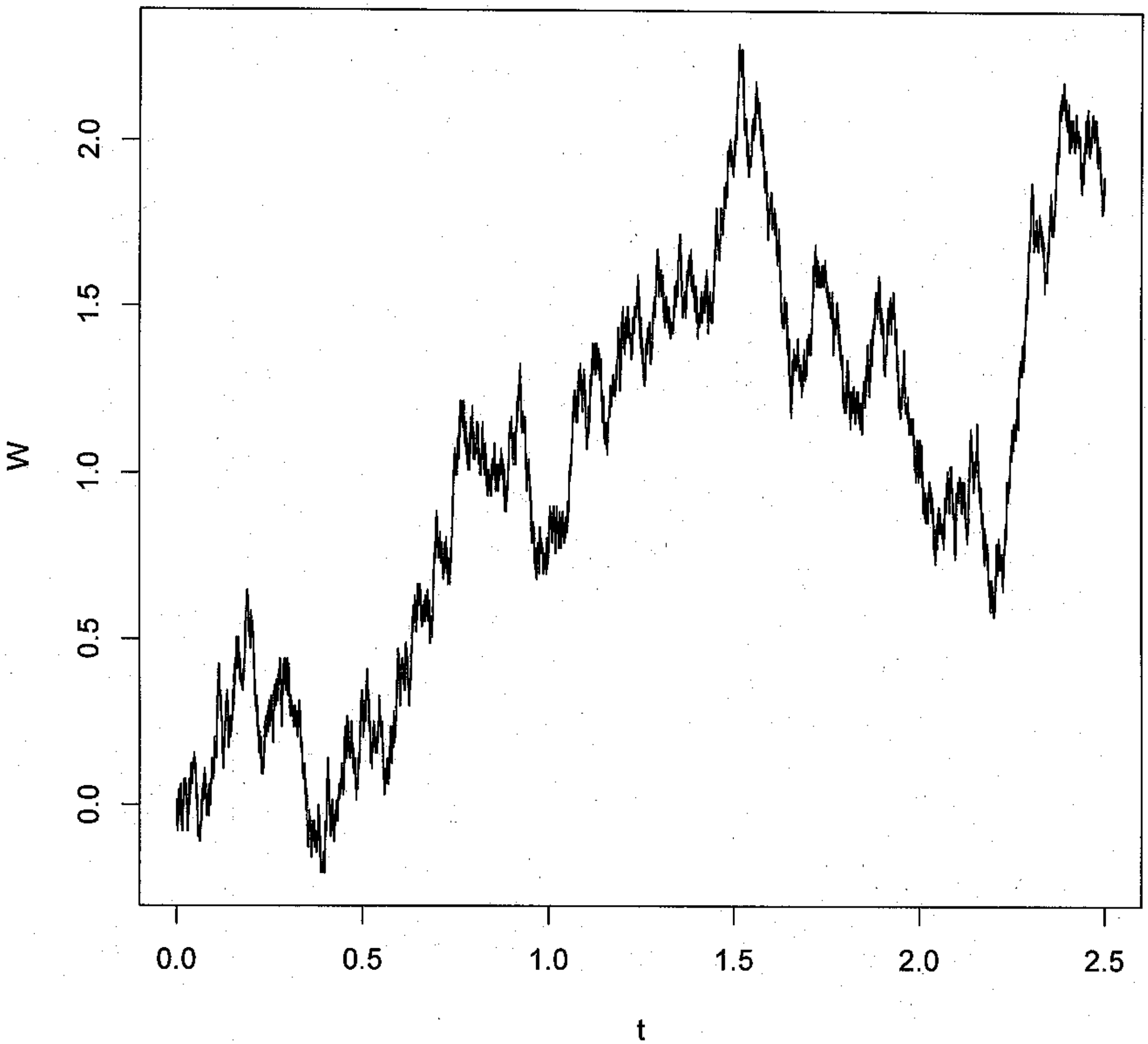
Exercises Chapter VI

VI.1

```
#Exercise VI.1

t0=2.5
N=10000
t=t0*c(0:N)/N
u=rbinom(N,1,0.5)
z=2*u-1
S=rep(0,N+1)
for (i in 1:N){
S[i+1]=S[i]+z[i]
}
W=S*sqrt(t0/N)
plot(t,W,type="l")
```

code to generate the Brownian motion on $[0, 2.5]$.


0.0
-0.5
-1.0
-1.5
-2.0
-2.5
W
0.0
0.5
1.0
1.5
2.0
2.5
t



W
0.0
0.5
1.0
1.5
2.0
2.5
t

VI.2 $\mathbb{E}(B_5 B_8) = \mathbb{E}(B_5(B_8 - B_5) + B_5^2)$

$= \mathbb{E}(B_5)\mathbb{E}(B_8 - B_5) + \mathbb{E}(B_5^2)$

$= 0 \cdot 0 + 5$ since $B_5 \sim N(0,5)$ ind of
$B_8 - B_5 \sim N(0,3)$

$= 5$

$Var(B_5 B_8) = \mathbb{E}(B_5^2 B_8^2) - 5^2$

$\mathbb{E}(B_5^2 B_8^2) = \mathbb{E}(B_5^2(B_8 - B_5 + B_5)^2)$

$= \mathbb{E}(B_5^2(B_8 - B_5)^2) + 2\mathbb{E}(B_5^3(B_8 - B_5)) + \mathbb{E}(B_5^4)$

$= \mathbb{E}(B_5^2)\mathbb{E}(B_8 - B_5)^2 + 2\mathbb{E}(B_5^3)\mathbb{E}(B_8 - B_5) + \mathbb{E}(B_5^4)$

$= 5 \cdot 3 + 2 \cdot 0 \cdot 0 + 5^2 \cdot 3 = 15 + 75 = 90$

since B_5, $B_8 - B_5$ are independent and
$\mathbb{E}(B_5^3) = \mathbb{E}(\sqrt{5}Z)^3 = 5^{3/2}\mathbb{E}(Z^3) = 0$
$\mathbb{E}(B_5^4) = \mathbb{E}((\sqrt{5}Z)^4) = 5^2\mathbb{E}(Z^4) = 5^2 \cdot 3$
where $Z \sim N(0,1)$

$\therefore Var(B_5 B_8) = 90 - 5^2 = 65$

VI.3 (a) $X_t = B_{t+\alpha} - B_\alpha$ then

1. $X_0 = B_\alpha - B_\alpha = 0$
2. $X_{t+s} - X_t = B_{t+s+\alpha} - B_\alpha - B_{t+\alpha} + B_\alpha$
$= B_{t+s+\alpha} - B_{t+\alpha}$
and so increments of the process $\{X_t\}$ are just increments of the $\{B_t\}$ process and so are mut. stat. ind.
3. $X_{t+s} - X_t = B_{t+s+\alpha} - B_{t+\alpha} \sim N(0, s)$
4. $Cov(X_s, X_t) = Cov(B_{s+\alpha} - B_\alpha, B_{t+\alpha} - B_\alpha)$
$= E((B_{s+\alpha} - B_\alpha)(B_{t+\alpha} - B_\alpha))$
$= E((B_{s+\alpha} - B_\alpha)B_{t+\alpha}) - E((B_{s+\alpha} - B_\alpha)B_\alpha)$
$\overset{s \le t}{=} E(B_{s+\alpha}B_{t+\alpha}) - E(B_\alpha B_{t+\alpha}) - E(B_{s+\alpha} - B_\alpha)E(B_\alpha)$
$= Cov(B_{s+\alpha}, B_{t+\alpha}) - Cov(B_\alpha, B_{t+\alpha}) - 0$
$= \min(s+\alpha, t+\alpha) - \min(\alpha, t+\alpha)$
$= \min(s+\alpha, t+\alpha) - \alpha = \begin{cases} s+\alpha-\alpha = s & s \le t \\ t+\alpha-\alpha = t & s \ge t \end{cases}$
$= \min(s,t)$
5. Since $B_t(\omega)$ is a continuous function of t then $X_t(\omega) = B_{t+\alpha}(\omega) - B_\alpha$ is also a continuous function of t

$\therefore$ $\{X_t : t \ge 0\}$ is a Brownian motion.

(b) $Y_t = \alpha B_{t/\alpha^2}$ see Example VI.1

VI.4 $\quad Y_t = 2+3t+4B_t$

$E(Y_3 Y_5) = E\left((2+3\cdot 3+4B_3)(2+3\cdot 5+4B_5)\right)$

$= E\left((11+4B_3)(17+4B_5)\right)$

$= 11\cdot 17 + 11\cdot 4\, E(B_5) + 4\cdot 17\, E(B_3) + 4\cdot 4\, E(B_3 B_5)$

$= 181 + 0 + 0 + 16\, cov(B_3, B_5)$

$= 181 + 16\cdot 3 = 229$

VI.5 (a) $cov(B_6, B_8) = \min(6,8) = 6.$

(b) $CORR(B_6, B_8) = \dfrac{cov(B_6, B_8)}{\sqrt{Var(B_6)\, Var(B_8)}}$

$= \dfrac{6}{\sqrt{6\cdot 8}} = \sqrt{\dfrac{6}{8}} = \sqrt{\dfrac{3}{4}}$

(c) $\lim\limits_{h\to 0} E\left(\dfrac{B_{8+h}-B_8}{h}\right) = \lim\limits_{h\to 0} \dfrac{E(B_{8+h})-E(B_8)}{h} = 0$

(d) $\lim\limits_{h\to 0} E\left(\dfrac{(B_{8+h}-B_8)^2}{h}\right) = \lim\limits_{h\to 0} \dfrac{h}{h} = 1$

since $E(B_{8+h}-B_8)^2 = h$ as $B_{8+h}-B_8 \sim N(0,h)$

VI.6

Let T_1, T_2, ... be i.i.d. interarrival times distributed exponential(1)
so E(T_i) = 1 so we generate 50 interarrival times since E(S_50)=50 although we may need more to cover the interval [0,50]. Note that S_n ~ gamma(n,1).

```
T=rgamma(50,1,1)
S=sum(T)
S
```

```
> T=rgamma(50,1,1)
> S=sum(T)
> S
[1] 53.8949
```

so 50 interarrival times is enough to cover [0,50]

```
S=rep(0,51)
N=rep(0,51)
for (i in 1:50){
S[i+1]=S[i]+T[i]
N[i+1]=N[i]+1
}
S
N
```

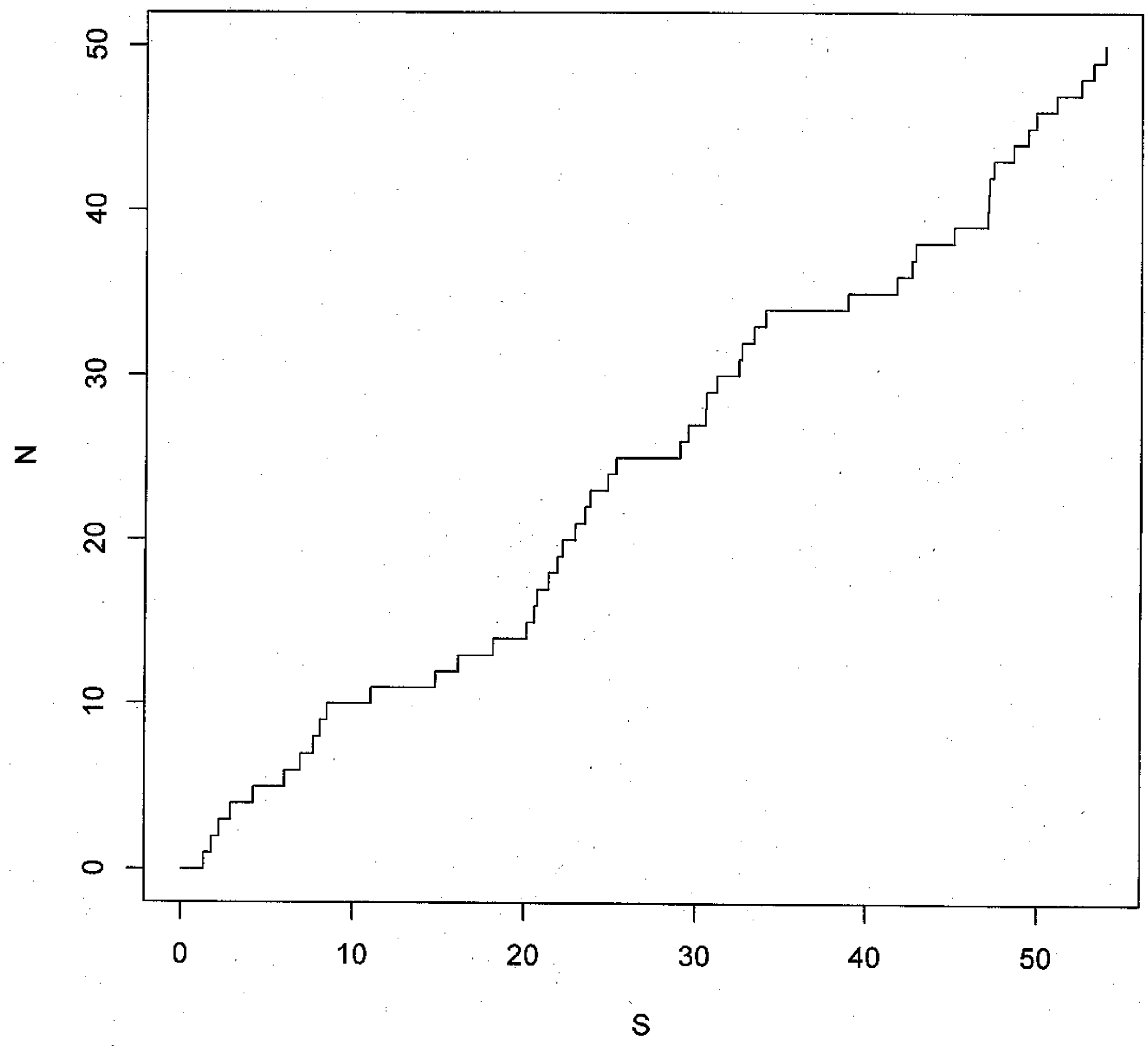
```
 [1]  0.000000  1.325109  1.757246  2.209095  2.866048  4.216820  6.036283
 [8]  6.957886  7.683156  8.085640  8.499416 11.061313 14.806002 16.123777
[15] 18.182220 20.105193 20.552615 20.719101 21.388874 21.900891 22.182885
[22] 22.934302 23.493441 23.782221 24.811451 25.311927 29.030378 29.506047
[29] 30.515188 30.579929 31.158321 32.484160 32.656942 33.342560 34.022096
[36] 38.845221 41.717798 42.580658 42.802418 45.050013 47.020079 47.061284
[43] 47.122314 47.358006 48.520410 49.388741 49.841573 51.047787 52.480698
[50] 53.202997 53.894904
> N
 [1]  0  1  2  3  4  5  6  7  8  9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
[26] 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49
[51] 50
```

arrival times

counts

```
plot(S,N,type="s")
```

plots stairsteps but the vertical lines aren't really part of the plot.



N
S
0
10
20
30
40
50

VI.7 If $X \sim \text{Poisson}(\lambda_1)$ ind. of $Y \sim \text{Poisson}(\lambda_2)$ then $X+Y \sim \text{Poisson}(\lambda_1+\lambda_2)$.

Therefore if $\{N_{i,t} : t \geq 0\}$ is a Poisson process of intensity λ_i and these stochastic processes are mut. stat. ind. then

(i) $N_{1,0} + \dots + N_{k,0} = 0 + \dots + 0 = 0$

(ii) $(N_{1,t+s} + \dots + N_{k,t+s}) - (N_{1,t} + \dots + N_{k,t})$

$$= (N_{1,t+s} - N_{1,t}) + (N_{2,t+s} - N_{2,t}) + \dots + (N_{k,t+s} - N_{k,t})$$

$$\sim \text{Poisson}(\lambda_1(t+s-t) + \dots + \lambda_k(t+s-t))$$

$$= \text{Poisson}((\lambda_1 + \dots + \lambda_k)s)$$

$\therefore \{\sum_{i=1}^{k} N_{i,t} : t \geq 0\}$ is a Poisson process of intensity $\lambda_1 + \dots + \lambda_k$.

VI.8 (a) $N_8 - N_5 \sim \text{Poisson}((8-5)\lambda) = \text{Poisson}(9)$

$\therefore E(N_8 - N_5) = 9$

$$P(N_8 - N_5 = 0) = \frac{9^0}{0!} e^{-9} = e^{-9}$$

(b) Recall $T_i = Y_1 + \cdots + Y_i$ where $Y_1, Y_2, \dots$ are i.i.d. exponential (λ) so $T_i \sim \text{gamma}_{rate}(i, \lambda)$.

Also

$$Y = \inf\{T_i : T_i > s\}$$

$$= \sum_{i=1}^{\infty} \left(I_{\{T_{i-1} \le s,\, T_i > s\}} \right) T_i$$

$$= \sum_{i=1}^{\infty} \left(I_{\{T_{i-1} \le s,\, Y_i > s - T_{i-1}\}} \right) T_i$$

Therefore,

$$E(Y) \overset{MCT}{=} \sum_{i=1}^{\infty} E\left(\left(I_{\{\ \}}\right)(T_{i-1} + Y_i) \right)$$

$$= E\left(I_{\{T_0 \le s,\, Y_1 > s - T_0\}} (T_0 + Y_1) \right) + \sum_{i=2}^{\infty} E\left(I_{\{T_{i-1} \le s,\, Y_i > s - T_{i-1}\}} (T_{i-1} + Y_i) \right) \quad \text{using } T_0 = 0$$

$$= \int_s^{\infty} y \lambda e^{-\lambda y} dy + \sum_{i=2}^{\infty} \int_0^s \left(\int_{s-t}^{\infty} \lambda e^{-\lambda y} dy \right) t \frac{\lambda^{i-1}}{\Gamma(i-1)} t^{i-2} e^{-\lambda t} dt$$

$$+ \sum_{i=2}^{\infty} \int_0^s \left(\int_{s-t}^{\infty} y \lambda e^{-\lambda y} dy \right) \frac{\lambda^{i-1}}{\Gamma(i-1)} t^{i-2} e^{-\lambda t} dt$$

$$= s e^{-\lambda s} + \frac{e^{-\lambda s}}{\lambda} + \sum_{i=2}^{\infty} \int_0^s e^{-\lambda(s-t)} \frac{\lambda^{i-1}}{\Gamma(i-1)} t^{i-1} e^{-\lambda t} dt$$

$$+ \sum_{i=2}^{\infty} \int_0^s \left((s-t) e^{-\lambda(s-t)} + \frac{e^{-\lambda(s-t)}}{\lambda} \right) \frac{\lambda^{i-1}}{\Gamma(i-1)} t^{i-2} e^{-\lambda t} dt$$

$$= s e^{-\lambda s} + \frac{e^{-\lambda s}}{\lambda} + s e^{-\lambda s} \sum_{i=2}^{\infty} \frac{(\lambda s)^{i-1}}{(i-1)!} + \frac{e^{-\lambda s}}{\lambda} \sum_{i=2}^{\infty} \frac{(\lambda s)^{i-1}}{(i-1)!}$$

$$= 5 e^{-\lambda 5} \sum_{i=0}^{\infty} \frac{(\lambda 5)^i}{i!} + \frac{e^{-\lambda 5}}{\lambda} \sum_{i=0}^{\infty} \frac{(\lambda 5)^i}{i!}$$

$$= 5 e^{-\lambda 5} e^{\lambda 5} + \frac{e^{-\lambda 5} e^{\lambda 5}}{\lambda} = 5 + \frac{1}{\lambda} = 5\,1/3$$

This is the hard solution. An easier (intuitive) solution is to remember the memory-less property of the exponential and so whatever $Y = T_1$ is $E(Y) = 5 + E(Y_1) = 5 + 1/3 = 5\,1/3$ but this isn't "rigourous".

VI.9 $E(XY) = E\big((N_6 - N_1)(N_5 - N_3)\big)$

$$= E\big((N_6 - N_5 + N_5 - N_3 + N_3 - N_1)(N_5 - N_3)\big)$$

$$= E\big((N_6 - N_5)(N_5 - N_3)\big) + E\big((N_5 - N_3)^2\big) + E\big((N_3 - N_1)(N_5 - N_3)\big)$$

and using independent increments and $N_{t+s} - N_t \sim \text{Poisson}(\lambda s)$, $E(N_{t+s} - N_t) = \lambda s$, $Var(N_{t+s} - N_t) = \lambda s$

$$= E(N_6 - N_5)E(N_5 - N_3) + E(N_5 - N_3)^2 + E(N_3 - N_1)E(N_5 - N_3)$$

$$= (\lambda(2\lambda)) + (2\lambda + 4\lambda^2) + ((2\lambda)(2\lambda))$$

$$= 2\lambda + 8\lambda^2 + 4\lambda^2 = 2\lambda + 12\lambda^2$$

VI, 10 Note $T \sim$ exponential $(3/2)$ independent of $U \sim$ exponential $(3/3)$ = exponential (1) based on results concerning thinning.

(a) $E(U) = 1$

(b) $P(Z > x) = P(\min(T, U) > x) = P(T > x, U > x)$

$= P(T > x) P(U > x)$ by independence

$= e^{-(3/2)x} e^{-x} = e^{-(5/2)x}$

(c) By (b) $Z \sim$ exponential$(5/2)$ so

$E(Z) = 2/5$

VI.11 $G = \begin{pmatrix} -3 & 3 & 0 \\ 1 & -2 & 1 \\ 0 & 4 & -4 \end{pmatrix} \therefore \hat{P} = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}$

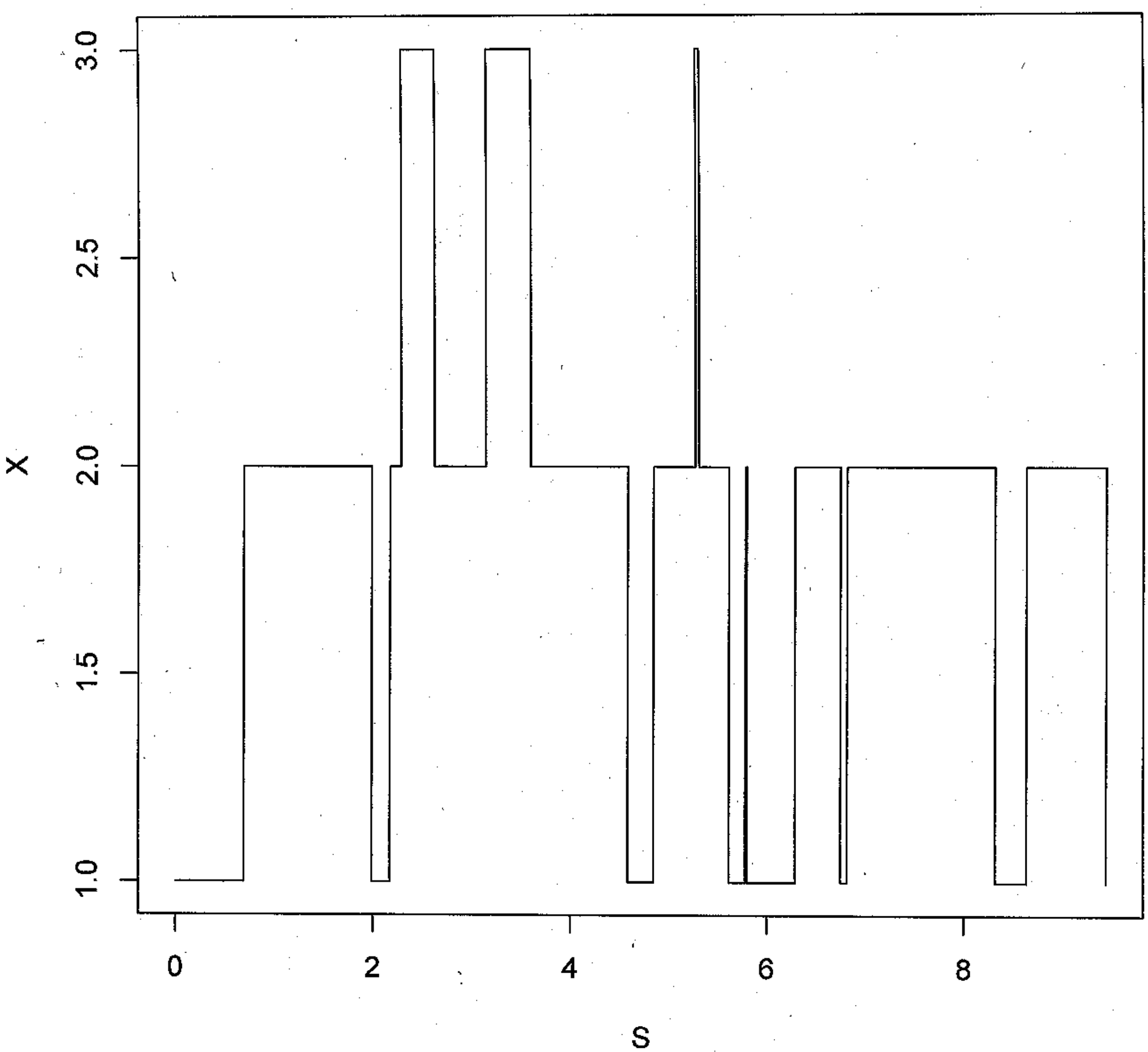

```
# intensities for the holding times
lambda=c(3,2,4)
# transition probability matrix for the jump chain
P1=c(0,1,0)
P2=c(1/2,0,1/2)
P3=c(0,1,0)
P=rbind(P1,P2,P3)
P
# X holds the states, T holds the holding times
#Start the process at X_0=1
X=rep(1,21)
T=rep(0,21)

for (i in 2:21){
rindex=X[i-1]
T[i-1]=rgamma(1,1,lambda[rindex])
X[i]=sample(c(1,2,3),size=1,prob=c(P[rindex,1],P[rindex,2],P[rindex,3]))
}

#S holds transition times
S=rep(0,21)
for ( i in 2:21){
S[i]=S[i-1]+T[i-1]
}
cbind(T,S,X)
               T         S X
 [1,] 0.69626276 0.0000000 1
 [2,] 1.29235776 0.6962628 2
 [3,] 0.18354204 1.9886205 1
 [4,] 0.10861937 2.1721626 2
 [5,] 0.33806014 2.2807819 3
 [6,] 0.52337388 2.6188421 2
 [7,] 0.45382068 3.1422159 3
 [8,] 0.98897450 3.5960366 2
 [9,] 0.26100938 4.5850111 1
[10,] 0.42244201 4.8460205 2
[11,] 0.03783394 5.2684625 3
[12,] 0.30683282 5.3062965 2
[13,] 0.16858974 5.6131293 1
[14,] 0.01480924 5.7817190 2
[15,] 0.49143034 5.7965283 1
[16,] 0.45882501 6.2879586 2
[17,] 0.06803041 6.7467836 1
[18,] 1.50722990 6.8148140 2
[19,] 0.31839075 8.3220439 1
[20,] 0.81200311 8.6404347 2
[21,] 0.00000000 9.4524378 1

plot(S,X,type="s")
```

T[i] = holding time for (i-1)st state.

S[i] = time of transition to next state.

X[i] = state at time S[i]

vertical lines are not part of the plot; just the horizontal lines.



VI.12 $P(t) = \begin{pmatrix} 1-7t & 7t & 0 \\ 0 & 1-3t & 3t \\ t & 2t & 1-3t \end{pmatrix} + o(t)$

$$g_{11} = \lim_{t \downarrow 0} \frac{1-7t-1}{t} + \frac{o(t)}{t} = -7$$

$$g_{22} = \lim_{t \downarrow 0} \frac{1-3t-1}{t} + \frac{o(t)}{t} = -3$$

$$g_{33} = \lim_{t \downarrow 0} \frac{1-3t-1}{t} + \frac{o(t)}{t} = -3$$

$$g_{12} = \lim_{t \downarrow 0} \frac{7t}{t} + \frac{o(t)}{t} = 7, \quad g_{13} = \lim_{t \downarrow 0} \frac{0}{t} + \frac{o(t)}{t} = 0$$

$$g_{21} = 0, \quad g_{23} = \lim_{t \downarrow 0} \frac{3t}{t} + \frac{o(t)}{t} = 3$$

$$g_{31} = \lim_{t \downarrow 0} \frac{t}{t} + \frac{o(t)}{t} = 1, \quad g_{32} = \lim_{t \downarrow 0} \frac{2t}{t} + \frac{o(t)}{t} = 2$$

VI.13

VI.13 $G = \begin{pmatrix} -7 & 7 \\ 3 & -3 \end{pmatrix}$

(a) $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = a_1 \begin{pmatrix} 3 \\ 7 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ initial dist. for starting at state 2

$\therefore \quad 3a_1 + a_2 = 0 \qquad a_1 = -a_2/3$

$7a_1 - a_2 = 1 \qquad -7a_2/3 - a_2 = 1$

or $\frac{10}{3} a_2 = -1$

$\therefore \quad a_2 = -\frac{3}{10}, \; a_1 = \frac{1}{10}$

$$\therefore \; p_{22}^{(t)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}' P^{(t)} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \left(\frac{1}{10} e^{0t} \begin{pmatrix} 3 \\ 7 \end{pmatrix}' - \frac{3}{10} e^{-10t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}' \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \left(\left(\frac{3}{10}, \frac{7}{10}\right) - \left(\frac{3}{10}, -\frac{3}{10}\right) e^{-10t} \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \left(\frac{3}{10}(1 - e^{-10t}), \; \frac{7}{10} + \frac{3}{10} e^{-10t} \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{7}{10} + \frac{3}{10} e^{-10t}$$

(b) $\lim_{t \to \infty} p_{22}^{(t)} = \frac{7}{10}$

VI.14

$$G = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & & \end{array} \\ \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \cdots \\ \mu & -\lambda-\mu & \lambda & 0 & \cdots \\ 0 & \mu & -\lambda-\mu & \lambda & 0 \cdots \\ 0 & 0 & \mu & & \\ \vdots & \vdots & \vdots & & \end{pmatrix} \end{array}$$

For $i=0$ $\quad \pi_0 g_{01} = \left(1-\frac{\lambda}{\mu}\right)\lambda = \frac{\lambda}{\mu}\left(1-\frac{\lambda}{\mu}\right)\mu = \pi_1 g_{10}$

$i=1, j=2$ $\quad \pi_1 g_{12} = \left(\frac{\lambda}{\mu}\right)\left(1-\frac{\lambda}{\mu}\right)\lambda = \left(\frac{\lambda}{\mu}\right)^2\left(1-\frac{\lambda}{\mu}\right)\mu = \pi_2 g_{21}$

$j \geq 3$ $\quad \pi_1 g_{1j} = \pi_1 0 = \pi_j g_{j1}$

and the remainder follow similarly.

VI.15 Since $\lambda = 3 < \mu = 5$ then

$$\lim_{t\to\infty} P(Q_t = 2) = \pi_2 = \left(\frac{\lambda}{\mu}\right)^2\left(1-\frac{\lambda}{\mu}\right)$$

$$= \left(\frac{3}{5}\right)^2\left(1-\frac{3}{5}\right) = \frac{18}{125}$$

VI.16 (a) $E(Y_i) = 5$ and so by the Elementary Renewal Theorem

$N_t/t \xrightarrow{wp1} 1/\mu = 1/5.$

(b) $E(\#\{n \geq 1 : 1234 < T_n < 1236\})$

$= E(N_{1234+2} - N_{1234})$ by the Blackwell Renewal Thm (BRT)

$= 2/5$

(c) $P(\exists n \geq 1 : 1234 < T_n < 1236)$ by Corollary to BRT

$= 2/5$

VI.17 $\mu = E(Y_1) = \int_0^4 y\left(\frac{y^3}{64}\right) dy$

$= \frac{1}{64}\int_0^4 y^4 \, dy = \frac{1}{64}\frac{1}{5} y^5 \Big|_0^4 = \frac{4^5}{4^3}\frac{1}{5} = \frac{16}{5}$

$\therefore$ by the BRT with $h = 3$

$E(\#\{n \geq 1 : 12345 < T_n < 12348\})$

$\approx 3/(16/5) = 15/16$

VI.18 $P(x, \cdot)$ is the Uniform$(0,1)$ probability measure.

(a) Given the chain is in state x the next state is always in $[0,1]$. Therefore if we take ϕ to be volume measure restricted to $[0,1]$ the chain is ϕ-irreducible.

(b) Yes the chain is aperiodic as there is no decomposition of $[0,1]$ into mutually disjoint subsets s.t. the chain cycles between the subsets.

(c) Put $\bar{\pi}$ be the $U(0,1)$ prob. measure.

Then $\int_0^1 P_x(A)\,\bar{\pi}(dx) = \int_0^1 \bar{\pi}(A)\,\bar{\pi}(dx)$ $= \bar{\pi}(A)\int_0^1 \bar{\pi}(dx) = \bar{\pi}(A)$ and so $\bar{\pi}$ is stationary.

(d) Since the chain is ϕ-irreducible and aperiodic, the General Convergence Theorem states that $\lim_{n\to\infty} P_x(X_n \in A) = \pi(A)$ $\forall A \in \mathcal{B}' \cap [0,1]$.

VI.19 Here we want to sample from the distribution with density proportional to $c_1 \exp\{-5x^2\}$ (which is a $N(0, 1/10)$ distribution). The proposal q is symmetric, and $q(x,y) > 0$ whenever $x \neq y$. The Metropolis algorithm proceeds as follows.

1. given $X_{n-1} = s$ generate $Y_n \sim q(s,\cdot)$ and $q(s,\cdot)$ is the density of a $N(x, 1/6)$ and independently generate $U_n \sim U(0,1)$.

2. put $X_n = \begin{cases} Y_n & \text{if } U_n \le \dfrac{\pi(Y_n)}{\pi(X_{n-1})} \\ X_{n-1} & \text{otherwise} \end{cases}$ where

$$\frac{\pi(Y_n)}{\pi(X_{n-1})} = \exp\{-5Y_n^2 + 5X_{n-1}\}$$