Edgeworth expansion for sample means

Assume $Y = (Y_1, \ldots, Y_n)$, where $Y_1, \ldots, Y_n$ are independent, identically distributed random variables, from a density function $f(y)$ with moment generating function $M_Y(t) = Ee^{tY}$ and cumulant generating function $K_Y(t) = \log M_Y(t)$. Let $S_n = Y_1 + \ldots + Y_n$ and $S_n^* = (S_n - n\mu)/\sqrt{n}\sigma$ where $\mu = EY$ and $\sigma^2 = varY$. Let ρ_3 and ρ_4 be the 3rd and 4th standardized cumulants: $\rho_3 = E(Y-\mu)^3/\sigma^3$, $\rho_4 = E(Y-\mu)^4/\sigma^4 - 3$.

The Edgeworth expansion for the distribution of the sample mean can be expressed in the following equivalent ways:

1.

$$f_{S_n^*}(x) = \phi(x)\{1 + \frac{\rho_3}{6\sqrt{n}}H_3(x) + \frac{\rho_4}{24n}H_4(x) + \frac{\rho_3^2}{72n}H_6(x)\} + O(n^{-3/2})$$

$$F_{S_n^*}(x) = \Phi(x) - \phi(x)\{\frac{\rho_3}{6\sqrt{n}}H_2(x) + \frac{\rho_4}{24n}H_3(x) + \frac{\rho_3^2}{72n}H_5(x)\} + O(n^{-3/2})$$

2.

$$f_{S_n}(z) = \frac{1}{\sqrt{n}\sigma}\phi(\frac{z-n\mu}{\sqrt{n}\sigma})\{1+\frac{\rho_3}{6\sqrt{n}}H_3(\frac{z-n\mu}{\sqrt{n}\sigma})+\frac{\rho_4}{24n}H_4(\frac{z-n\mu}{\sqrt{n}\sigma})+\frac{\rho_3^2}{72n}H_6(\frac{z-n\mu}{\sqrt{n}\sigma})\}+O(n^{-3/2})$$

3. Let $\phi(z; a, b^2) = \frac{1}{\sqrt{b}}\phi(\frac{z-a}{b})$, $H_r(z; a, b^2) = (-1)^r\frac{\phi^{(r)}(z;a,b^2)}{\phi(z;a,b^2)}$, and let $\kappa_r^n = \kappa_r(S_n)$ be the rth cumulant of S_n.

$$f_{S_n}(s) = \phi(z; n\mu, n\sigma^2)\{1+\frac{\kappa_3^n}{6}H_3(z; n\mu, n\sigma^2)+\frac{\kappa_4^n}{24}H_4(z; n\mu, n\sigma^2)+\frac{\kappa_6^n}{72}H_6(z; n\mu, n\sigma^2)\}+O(n^{-3/2})$$