UNIVERSITY OF TORONTO

Faculty of Arts and Science

Term Test, October 19th, 2010

ACT 460 / STA 2502

DURATION - 120 **minutes**

EXAMINER: Prof. S. Jaimungal

COURSE CODE (**circle one**): ACT 460 STA 2502

LAST NAME: ____

FIRST NAME: ____

STUDENT #: ____

Each question is worth 10 points

– NOT ALL QUESTIONS ARE OF THE SAME DIFFICULTY .

Please write clearly!

1 *[10]*	2 *[10]*	3 *[10]*	4 *[10]*	5 *[10]*	6 *[10]*	Total *[60]*

1. *[10]* Please indicate true or false. **no explanations required**

   -1 **for incorrect answer,** $+2$ **for correct answer,** 0 **for blank answer** .

   (a) [T] [F]

   In an arbitrage-free economy, there exists a unique risk-neutral measure.

   e.g. trinomial model with two traded assets

   (b) [T] [F]

   The price of a put option always decreases with increasing volatility.

   prices increase with vol.

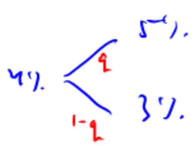
   (c) [T] [F]

   In the one-period binomial model, if an economy has two distinct traded assets, then any contingent claim can be replicated.

b/c 2-d linear system with 2 unknowns can always be solved.

   (d) [T] [F]

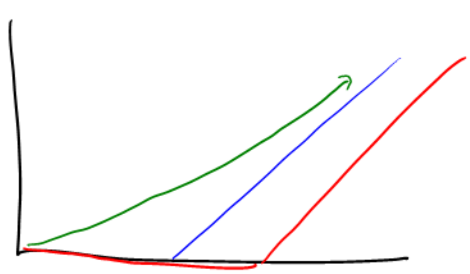
   If the risk-free interest rate follows a multi-period binomial tree, then the risk-neutral branching probabilities are $\frac{1}{2}$.



$q \in (0,1)$ may not equal $\frac{1}{2}$.

   (e) [T] [F]

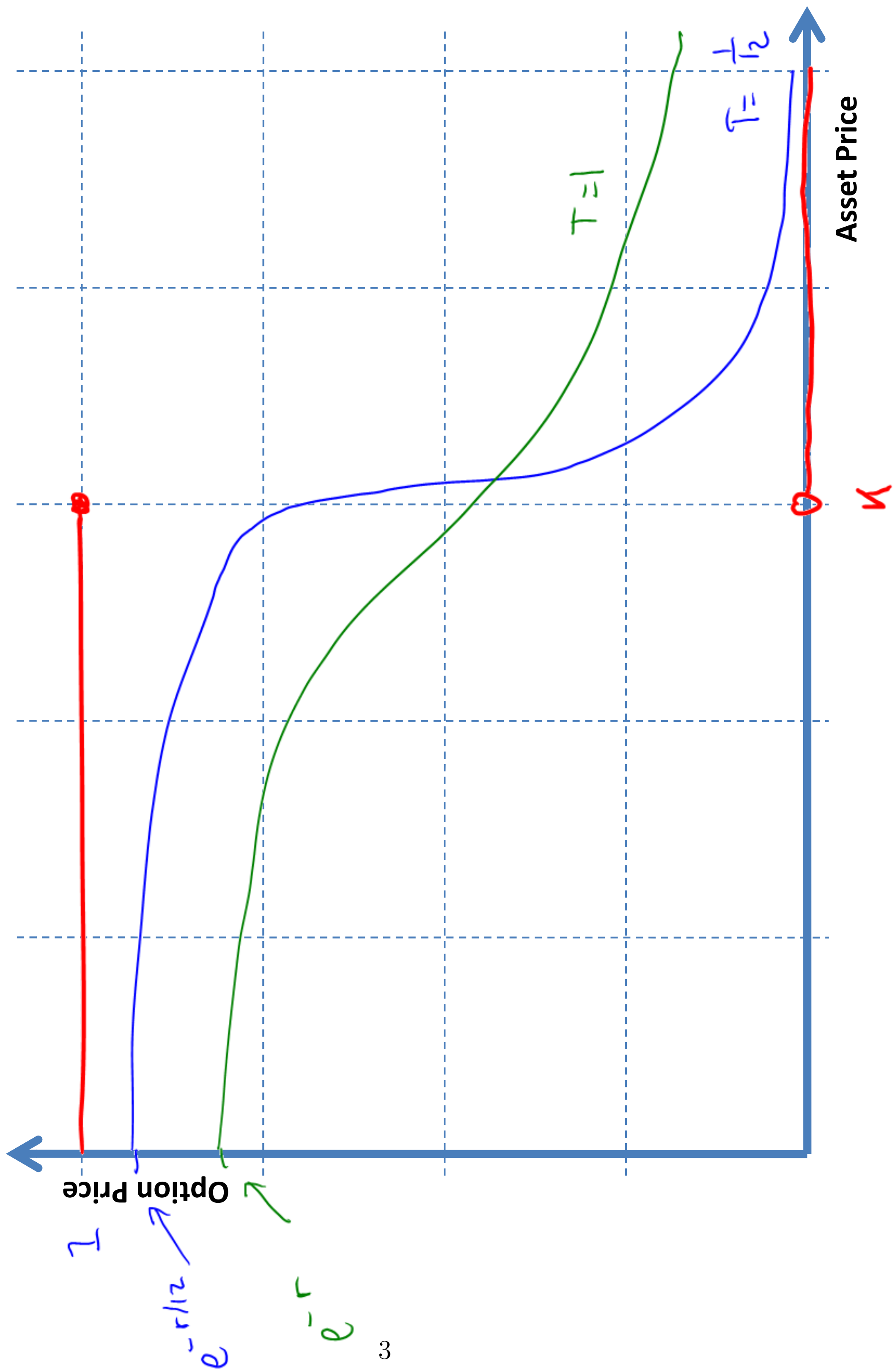
   The price of a T-maturity call option, on an asset with no dividends with strike K, approaches the line $S - Ke^{-rT}$ **from below** as the spot price increases to infinity.



2. Sketch the option price as a function of the current spot-level for maturities of $T = 0$, $T = 1$ month and $T = 1$ year for

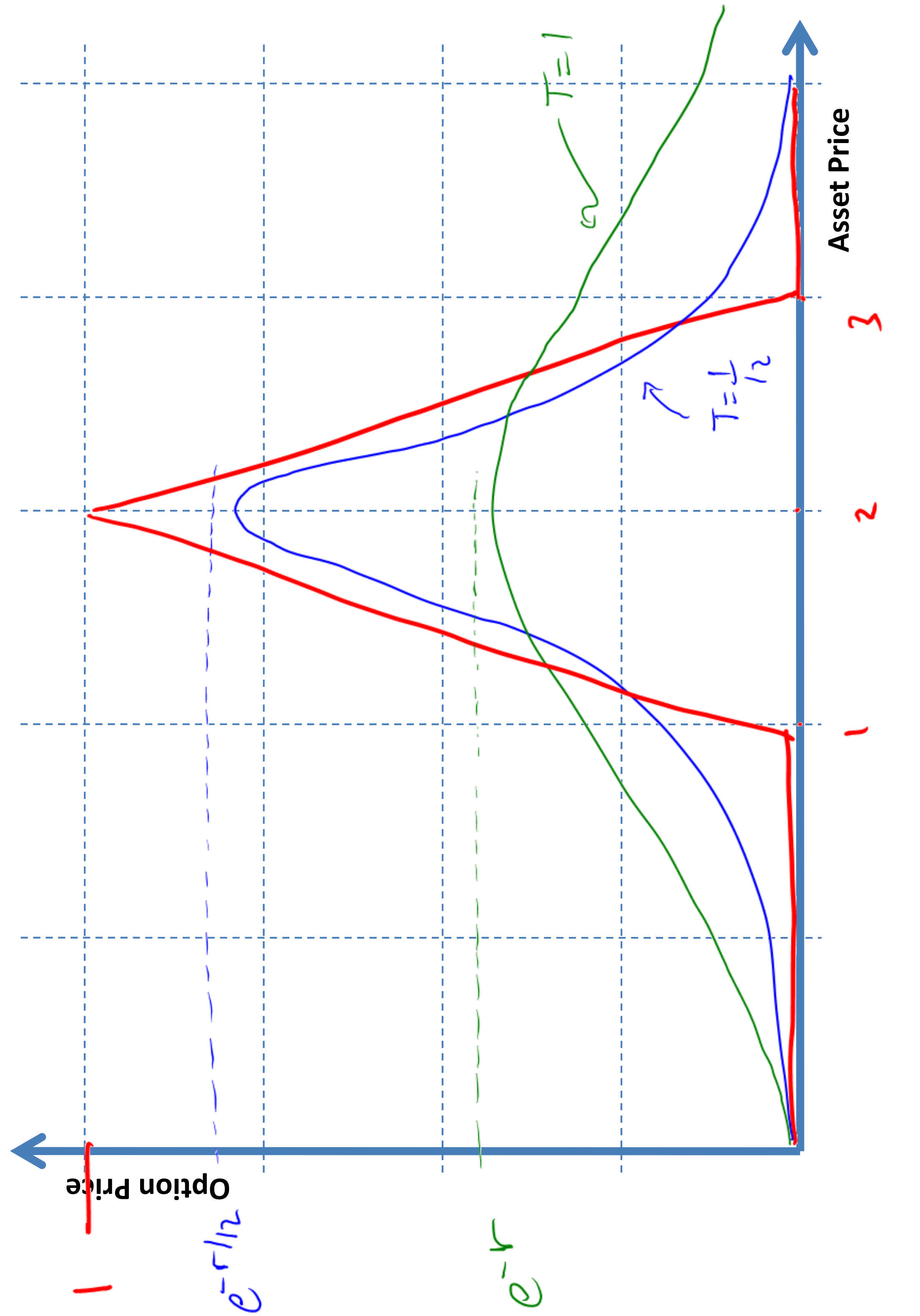
(a) *[5]* digital put option (which pays 1 if $S < K$ and 0 otherwise).

[draw the three curves on the same graph, clearly label them and any interesting points.]



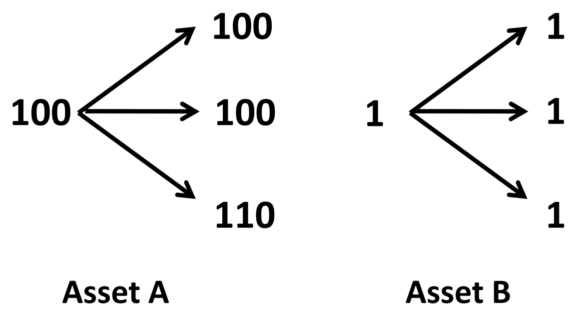
(b) *[5]* A portfolio consisting of 1 long call struck at \$1, 2 short calls struck at \$2 and 1 long call struck at \$3.

[draw the three curves on the same graph, clearly label them and any interesting points..]



3. *[10]* Consider an economy with the two traded assets below. State the formal conditions of an arbitrage strategy and construct one for this economy.

[As usual, all real-world branching probabilities are strictly positive.]



An arbitrage strategy with value V_t is one s.t.

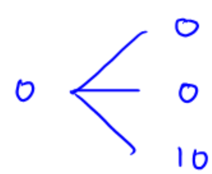
a) $V_0 = 0$

b) $\exists$ a t s.t. i) $\mathbb{P}(V_t \geq 0) = 1$

ii) $\mathbb{P}(V_t > 0) > 0$

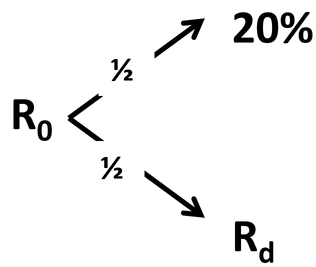
take 1 unit of Asset A

-100 units of Asset B



So all conditions of an arbitrage strategy are satisfied

4. (a) *[6]* Consider the interest rate tree shown in the diagram below – each time step is 1-year. The rates correspond to effective discounting – e.g. discounting over the first period is $1/(1+R_0)$. The probabilities shown are risk-neutral probabilities.



A one-year zero coupon bond on a notional of \$100 costs \$91.1121. As well, a 2-year coupon bearing bond with coupons of \$10 paid every year and notional of \$100 is valued at par (i.e. is valued at \$100). Calibrate this model to the market prices, i.e. determine R_0 and R_d such that the market prices are equal to the model prices.

$$P_0(1) = 91.1121 = \frac{100}{1+R_0} \Rightarrow R_0 = 9.75\%$$

2-yr coupon bond price is

$$\frac{110}{1.20} + 10 = 101.67 \qquad \frac{110}{1+R_d} + 10$$

110, 110, 110, 110

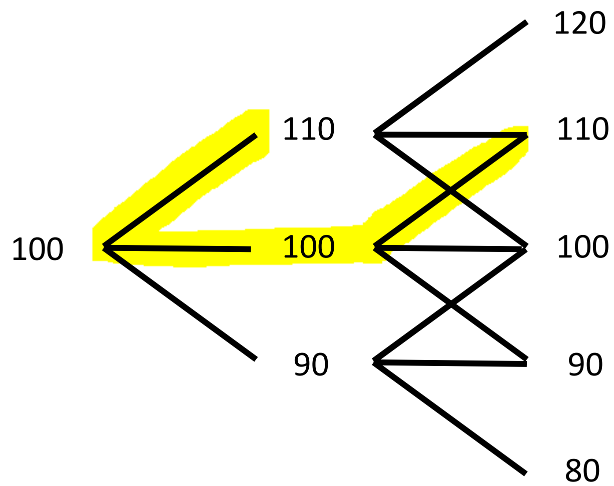
$$100 = \frac{1}{1+R_0}\left\{\frac{1}{2}\times 101.67 + \frac{1}{2}\times\left(\frac{110}{1+R_d} + 10\right)\right\}$$

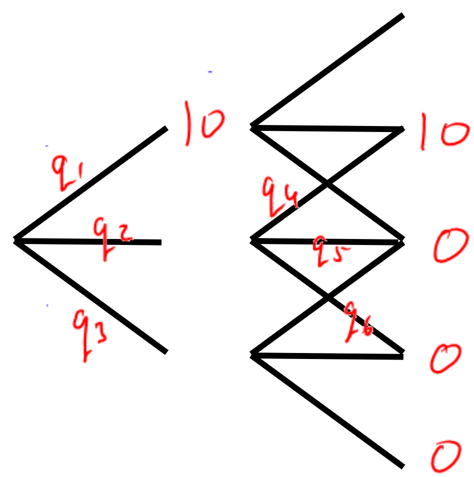
$$\Rightarrow \frac{110}{1+R_d} = 200(1+R_0) - 111.67 = 107.83$$

$$\Rightarrow R_d = 2\%$$

(b) *[4]* You are given the following model (on the left) for a stock price:

[the extra tree on the right is include just for your convenience.]





Consider a 2-period barrier option which pays 10 the instant the asset touches the level 110, otherwise it pays 0. Assume interest rates are 0. Find the no-arbitrage bounds on the value of the Barrier option.

Only the highlighted paths result in payments of 10

(or use payoff in tree to the right)

and valuation is $V_0 = 10\,(q_1 + q_2\, q_4)$

now need ranges for q_1, q_2 & q_4 ...

a

$$\left.\begin{aligned} q_1\,110 + q_2\,100 + q_3\,90 &= 100 \\ q_1 + q_2 + q_3 &= 1 \end{aligned}\right\} \Rightarrow \quad q_1 = q_3 \quad \text{and} \quad q_2 = 1 - 2q_1$$

so, then $q_1 > 0$, $q_2 > 0$ & $q_3 > 0 \Rightarrow 0 < q_1 < \frac{1}{2}$

b

similarly: $q_4 = q_6$ and $q_5 = 1 - 2q_4$

$$0 < q_4 < \tfrac{1}{2}$$

so $V_0 = 10\,(q_1 + (1-2q_1)\, q_4)$

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now this is minimized when $q_1 = q_4 = 0 \Rightarrow V_o > 0$

and maximized when $q_1 = q_4 = \frac{1}{2} \Rightarrow V_o < 5$

or $q_1 = 0$
$q_4 = \frac{1}{2}$

or $q_1 = \frac{1}{2}$
$q_4 = 0$

5. Assume an equity price S_t is modeled as in the Black-Scholes model (i.e. the limiting case of the CRR model as $\Delta t \downarrow 0$ and interest rates are constant at r). For each of the following, write your answers terms of $\Phi(x) \triangleq \mathbb{Q}(Z < x)$ where Z is a standard normal random variable under the risk-neutral measure $\mathbb{Q}$.

   (a) *[5]* Derive an expression for the $(t = 0)$ price of an option with T-maturity payoff

$$\varphi = (S_T)^2 \, \mathbb{I}_{S_T > K} \, .$$

   Here K is a constant and, as usual, $\mathbb{I}_\omega$ is the indicator function of the event ω, i.e. equals 1 if ω occurs and 0 if ω otherwise.

$$S_T \overset{d}{=} S_0 \, e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T} Z} \, , \quad Z \underset{\mathbb{Q}}{\sim} N(0,1)$$

$$V_0 = e^{-rT} \, \mathbb{E}^{\mathbb{Q}}\left[(S_T)^2 \, \mathbb{1}_{S_T > K} \right]$$

$$= e^{-rT} \int_{-\infty}^{\infty} \left(S_0 \, e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T} z} \right)^2 \mathbb{1}_{z > z^*} \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}} \, dz$$

where $z^* = - \dfrac{\ln(S_0/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$

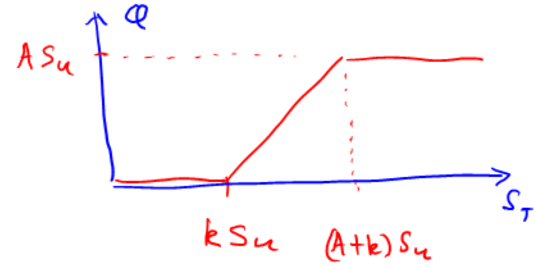
$$= e^{-rT} \, S_0^2 \, e^{2(r - \frac{1}{2}\sigma^2)T} \int_{z^*}^{\infty} e^{2\sigma\sqrt{T} z - \frac{1}{2}z^2} \frac{dz}{\sqrt{2\pi}}$$

$$= S_0^2 \, e^{(r - \sigma^2)T} \int_{z^*}^{\infty} e^{-\frac{1}{2}(z - 2\sigma\sqrt{T})^2 + \frac{1}{2}(2\sigma\sqrt{T})^2} \frac{dz}{\sqrt{2\pi}}$$

$$= S_0^2 \, e^{(r + \sigma^2)T} \, \Phi\left(-z^* + 2\sigma\sqrt{T} \right)$$

(b) *[5]* Derive an expression for the ($t = 0$) price of a forward starting option with T-payoff

$$\varphi = \min(S_T - k\, S_U\, ,\, A\, S_U)\ .$$



Here, $0 < U < T$, $k > 0$ and $A > 0$ are constants.

This is a long call struck @ kS_u
short call struck @ $(A+k)S_u$

so $V_0 = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}\left[\,(S_T - kS_u)_+ - (S_T - (A+k)S_u)_+\,\right]$

let's value

$$U_0 = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}\left[(S_T - \alpha S_u)_+\right]$$

$$= e^{-rT}\,\mathbb{E}^{\mathbb{Q}}\Big[\underbrace{\mathbb{E}^{\mathbb{Q}}\left[(S_T - \alpha S_u)_+ \mid S_u\right]}_{e^{r(T-u)}\left(S_u\,\Phi(d_+) - \alpha S_u\, e^{-r(T-u)}\,\Phi(d_-)\right)}\Big]$$

$$= e^{r(T-u)}\,\beta(\alpha)\,S_u,$$

where: $\beta(\alpha) = \Phi(d_+) - \alpha\, e^{-r(T-u)}\,\Phi(d_-) = \text{const.}$

and $d_\pm = \dfrac{\ln(S_u/\alpha S_u) + (r \pm \frac{1}{2}\sigma^2)(T-u)}{\sigma\sqrt{T-u}} = \text{const.}$

$$\Rightarrow U_0 = \beta(\alpha)\,e^{-ru}\,\mathbb{E}^{\mathbb{Q}}[S_u] = \beta(\alpha)\,S_0$$

$$\Rightarrow \underline{V_0 = \left(\beta(k) - \beta(A+k)\right)S_0}$$

6. Consider the modified CRR model of stock prices

$$S_{n\Delta t} = S_{(n-1)\Delta t} \exp\left\{(\mu - \tfrac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\, x_n\right\}$$

where $x_1, x_2, \ldots$ are iid r.v. with $\mathbb{P}(x_1 = +1) = p$ and $\mathbb{P}(x_1 = -1) = 1 - p$. Interest rates are constant so that the money-market account M_t evolves as

$$M_{n\Delta t} = M_{(n-1)\Delta t} \exp\{r\Delta t\}\,.$$

(a) *[4]* Prove that, in this model, under the risk-neutral measure $\mathbb{Q}$, the up-branching probabilities as $\Delta t \downarrow 0$ are

$$q = \frac{1}{2}\left[1 + \frac{r-\mu}{\sigma}\sqrt{\Delta t}\right] + o(\sqrt{\Delta t})\,.$$

Recall that $o(\sqrt{\Delta t})$ is a term which goes to zero **faster than** $\sqrt{\Delta t}$.

$$S_{(n-1)\Delta t} = e^{-r\Delta t}\, \mathbb{E}^{\mathbb{Q}}[S_{n\Delta t}]$$

$$\Rightarrow \quad e^{r\Delta t} = q\, e^{(\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}} + (1-q)\, e^{(\mu - \frac{1}{2}\sigma^2)\Delta t - \sigma\sqrt{\Delta t}}$$

$$\Rightarrow \quad q = \frac{e^{r\Delta t} - e^{(\mu - \frac{1}{2}\sigma^2)\Delta t - \sigma\sqrt{\Delta t}}}{e^{(\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}} - e^{(\mu - \frac{1}{2}\sigma^2)\Delta t - \sigma\sqrt{\Delta t}}}$$

$$= \frac{e^{(r - \mu + \frac{1}{2}\sigma^2)\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$$

$$= \frac{(1 + (r - \mu + \frac{1}{2}\sigma^2)\Delta t) - (1 - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t) + o(\Delta t)}{(1 + \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t) - (1 - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t) + o(\Delta t)}$$

$$= \frac{\sigma\sqrt{\Delta t} + (r-\mu)\Delta t + o(\Delta t)}{2\sigma\sqrt{\Delta t} + o(\Delta t)}$$

$$= \frac{1}{2}\left(1 + \frac{r-\mu}{\sigma}\sqrt{\Delta t}\right) + o(\sqrt{\Delta t})$$

(b) *[6]* Prove that in the limit as $\Delta t \downarrow 0$, the joint distribution of the asset price at two points in time, T_1 and T_2 with $T_2 > T_1$, can be written as follows:

$$S_{T_1} \overset{d}{=} S_0\, e^{\hat{r}\,T_1+\sigma\sqrt{T_1}\,Z_1} \qquad \text{and} \qquad S_{T_2} \overset{d}{=} S_0\, e^{\hat{r}\,T_2+\sigma\sqrt{T_2}\,Z_2}$$

where, Z_1 and Z_2 are $\mathbb{Q}$-bivariate standard normal random variables with correlation

$$\rho = \sqrt{\frac{T_1}{T_2}},$$

and $\hat{r} = r - \frac{1}{2}\sigma^2$.

let $T_2 = (M+N)\Delta t$ & $T_1 = N\Delta t$

$$\Rightarrow \quad S_{T_1} = S_0 \exp\left\{ (\mu - \tfrac{1}{2}\sigma^2)\Delta t\, N + \underbrace{\sigma\sqrt{\Delta t}\sum_{n=1}^{N} x_n}_{X} \right\}$$

$$S_{T_2} = S_0 \exp\left\{ (\mu - \tfrac{1}{2}\sigma^2)\Delta t\,(M+N) + \underbrace{\sigma\sqrt{\Delta t}\sum_{n=1}^{M+N} x_n}_{Y} \right\}$$

$$\text{by CLT} \quad (X,Y) \xrightarrow[M,N \to +\infty]{} \mathcal{N}\left(\begin{pmatrix} m_x \\ m_y \end{pmatrix} ; \begin{pmatrix} \Omega_{xx} & \Omega_{xy} \\ \Omega_{yx} & \Omega_{yy} \end{pmatrix} \right)$$

so need m_x, m_y, Ω_{ij} ...

$$m_x = \sigma\sqrt{\Delta t}\; \mathbb{E}^{\mathbb{Q}}\Big[\sum_{n=1}^{N} x_n \Big] \overset{\text{since } x_n \text{ iid}}{=} \sigma\sqrt{\Delta t}\; N\, \mathbb{E}^{\mathbb{Q}}[x_1]$$

$$= \sigma\sqrt{\Delta t}\, N\,(2q-1) = (r-\mu)\,\Delta t\, N + o(\Delta t) \to (r-\mu)\,T_1$$

similarly,

$$m_y = \sigma\sqrt{\Delta t}\,(M+N)\,(2q-1) \longrightarrow (r-\mu)\,T_2$$

$$\Omega_{xx} = \sigma^2\Delta t\; \mathbb{V}^{\mathbb{Q}}\Big[\sum_{n=1}^{N} x_n \Big] \overset{\text{since } x_n \text{ iid}}{=} \sigma^2\,\Delta t\, N\; \mathbb{V}^{\mathbb{Q}}[x_1]$$

$$= \sigma^2\,\Delta t\, N\,\left(1-(2q-1)^2\right) = \sigma^2\, T_1 \left(1 - \left(\tfrac{r-\mu}{\sigma}\right)\sqrt{\Delta t} + o(\sqrt{\Delta t}) \right)$$

$$\to \sigma^2\, T_1$$

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similarly,

$$\Omega_{yy} \longrightarrow \sigma^2 T_2$$

these sums are independent since x_n iid

$$\Omega_{xy} = \sigma^2 \Delta t \ \mathbb{C}\left[\sum_{n=1}^{N} x_n , \sum_{n=1}^{M+N} x_n \right]$$

$$= \sigma^2 \Delta t \left(\mathbb{C}\left[\sum_{n=1}^{N} x_n , \sum_{n=1}^{N} x_n \right] + \mathbb{C}\left[\underbrace{\sum_{n=1}^{N} x_n , \sum_{n=N+1}^{M} x_n} \right] \right)$$

$$= \sigma^2 \Delta t \ N \ \mathbb{V}[x_1]$$

$$= \sigma^2 T_1 \left(1 - \frac{r-\mu}{\sigma} \sqrt{\Delta t} + o(\sqrt{\Delta t}) \right) \longrightarrow \sigma^2 T_1$$

so,
$$\begin{pmatrix} X \\ Y \end{pmatrix} \underset{\mathbb{Q}}{\sim} \mathcal{N}\left(\begin{pmatrix} (r-\mu) T_1 \\ (r-\mu) T_2 \end{pmatrix} ; \begin{pmatrix} \sigma^2 T_1 & \sigma^2 T_1 \\ \sigma^2 T_1 & \sigma^2 T_2 \end{pmatrix} \right)$$

$$\Rightarrow \quad X \overset{d}{=} (r-\mu) T_1 + \sigma\sqrt{T_1} \ Z_1$$
$$Y \overset{d}{=} (r-\mu) T_2 + \sigma\sqrt{T_2} \ Z_2$$

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} ; \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right) \quad \text{and} \quad \rho = \sqrt{\frac{\sigma^2 T_1}{\sigma^2 T_2}} = \sqrt{\frac{T_1}{T_2}}$$

so then,

$$S_{T_1} \overset{d}{=} S_0 \ e^{(\mu - \frac{1}{2}\sigma^2) T_1 + (r-\mu) T_1 + \sigma\sqrt{T_1} Z_1}$$
$$= S_0 \ e^{\hat{r} T_1 + \sigma\sqrt{T_1} Z_1}$$

$$S_{T_2} \overset{d}{=} S_0 \ e^{(\mu - \frac{1}{2}\sigma^2) T_2 + (r-\mu) T_2 + \sigma\sqrt{T_2} Z_2}$$
$$= S_0 \ e^{\hat{r} T_2 + \sigma\sqrt{T_2} Z_2}$$