Ho-Lee Continuous Limit

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Ho-Lee Model:

$$r_n = r_{n-1} + \theta_{n-1}\Delta t + \sigma\sqrt{\Delta t}\, x_n$$

$\llcorner$ iid ± 1 $\quad q = 1/2$

$0 \quad \Delta t \quad \cdots \qquad\qquad T = N\Delta t \qquad \Delta t \downarrow 0$

$(N \to \infty)$

$$r_T = r_0 + \sum_{n=1}^{N} \theta_{n-1}\Delta t + \sigma\sqrt{\Delta t}\sum_{n=1}^{N} x_n$$

$$\xrightarrow[N\to+\infty]{Q} \mathcal{N}(m; v)$$

by CLT

$$m = \mathbb{E}^{Q}[r_T] = r_0 + \sum_{n=1}^{N}\theta_{n-1}\Delta t \xrightarrow[N\to+\infty]{} r_0 + \int_0^T \theta_u\, du$$

$$v = \mathbb{V}^{Q}[r_T] = \mathbb{V}^{Q}\Big[\sigma\sqrt{\Delta t}\sum_{n=1}^{N} x_n\Big]$$

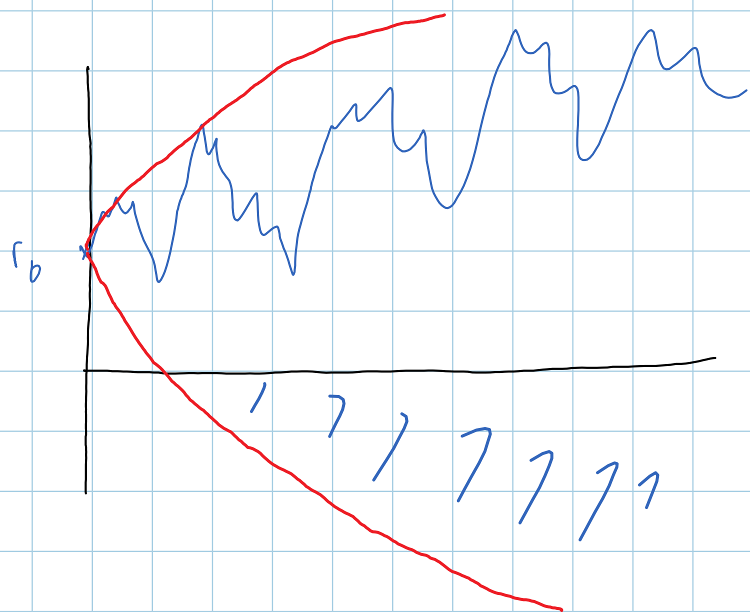
$$= \sigma^2\Delta t\sum_{n=1}^{N}\mathbb{V}^{Q}[x_n] = \sigma^2 . \Delta t . N$$

$\llcorner$ 1

$$= \sigma^2 T$$

$\therefore$ as $N\to+\infty$, $\boxed{r_T \overset{d}{=} r_0 + \int_0^T \theta_u\, du + \sigma\sqrt{T}\, Z, \quad Z \underset{Q}{\sim} \mathcal{N}(0,1).}$

$\therefore$ as $N \to +\infty$, $\boxed{r_T \overset{d}{=} r_0 + \int_0^T \theta_u \, du + \sigma \sqrt{T} \, Z, \quad Z \underset{\mathbb{Q}}{\sim} \mathcal{N}(0,1).}$



$\mathbb{P}(r_t < 0) = \ldots$

$$A \begin{cases} A e^{\sigma\sqrt{\Delta t}} \\ A e^{-\sigma\sqrt{\Delta t}} \end{cases} \qquad r \begin{cases} r + \sigma\sqrt{\Delta t} + \theta \Delta t \\ r - \sigma\sqrt{\Delta t} + \theta \Delta t \end{cases}$$

$$r \begin{cases} r e^{\theta \Delta t + \sigma\sqrt{\Delta t}} \\ r e^{\theta \Delta t - \sigma\sqrt{\Delta t}} \end{cases}$$

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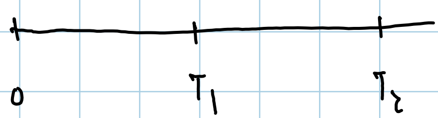
bond prices in the limit ...

$$P_0(T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r_0 \Delta t} \; e^{-r_1 \Delta t} \cdots e^{-r_{N-1} \Delta t} \; 1 \right]$$

$$P_0(T) = \mathbb{E}\left[e \quad e \quad \dots e \quad \right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[e^{-\sum_{n=1}^{N} r_{n-1}\,\Delta t} \right]$$

$$\to \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_u\,du} \right]$$



joint distribution (r_{T_1}, r_{T_2})

$\mathbb{E}^{\mathbb{Q}}[r_{T_1}]$, $\mathbb{E}^{\mathbb{Q}}[r_{T_2}]$

$\mathbb{V}^{\mathbb{Q}}[r_{T_1}]$, $\mathbb{V}^{\mathbb{Q}}[r_{T_2}]$

$\mathbb{C}[r_{T_1}, r_{T_2}] = ?$

to find $P_0(T)$ need dist of $\int_0^T r_u\,du$.

$$\sum_{n=1}^{N} r_{n-1}\,\Delta t = \sum_{n=1}^{N}\left(r_0 + \sum_{m=1}^{n-1} \theta_{m-1}\,\Delta t + \sum_{m=1}^{n-1} \sigma\sqrt{\Delta t}\, x_m \right)\Delta t$$

$$= r_0 T + \underbrace{\theta\,\Delta t^2 \underbrace{\sum_{n=1}^{N}(n-1)}_{\frac{(N-1)N}{2}}}_{\to \frac{\theta T^2}{2}} + \underbrace{\sum_{n=1}^{N}\sum_{m=1}^{n-1} x_m}_{A}\; \sigma(\Delta t)^{3/2}$$

$$\left(\int_0^T \int_0^u \theta_s\,ds\,du \right)$$

$$\begin{aligned} A = \; & 0 \\ & + x_1 \\ & + x_1 + x_2 \\ & + x_1 + x_2 + x_3 \\ & \;\vdots \end{aligned}$$

$$\vdots$$

$$+ \; x_1 + x_2 + x_3 + \cdots + x_{N-1}$$

$$(N-1)x_1 + (N-2)x_2 + \cdots + x_{N-1}$$

$$\mathbb{V}[A] = \sum_{n=1}^{N-1} (N-n)^2 \underbrace{\mathbb{V}[x_n]}_{\hookrightarrow 1} = \sum_{n=1}^{N-1} (N-n)^2 = \frac{N^3}{3} + O(N^2)$$

$$\boxed{\int_0^T r_u\,du \xrightarrow[N\to+\infty]{d,\mathbb{Q}} r_0 T + \int_0^T \int_0^u \theta_s\,ds\,du + \left(\frac{\sigma^2 T^3}{3}\right)^{1/2} Z \qquad \underset{\mathbb{Q}}{Z \sim N(0,1)}}$$

$$P_0(T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_u\,du} \right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[\exp\left\{ -r_0 T - \int_0^T \int_0^u \theta_s\,ds\,du - \left(\frac{\sigma^2 T^3}{3}\right)^{1/2} Z \right\} \right]$$

$$= \exp\left\{ -r_0 T - \int_0^T \int_0^u \theta_s\,ds\,du \right\} \underbrace{\mathbb{E}^{\mathbb{Q}}\left[e^{-\left(\frac{\sigma^2 T^3}{3}\right)^{1/2} Z} \right]}_{\hookrightarrow e^{\frac{\sigma^2 T^3}{6}}}$$

$$\boxed{P_0(T) = \exp\left\{ -r_0 T - \int_0^T \int_0^u \theta_s\,ds\,du + \frac{\sigma^2 T^3}{6} \right\}.}$$

$$\ln P_0^*(T) = -r_0 T - \int_0^T \int_0^u \theta_s \, ds \, du + \frac{\sigma^2 T^3}{6}$$

$$\partial_T \ln P_0^*(T) = -r_0 - \int_0^T \theta_s \, ds + \frac{\sigma^2 T^2}{2}$$

$$\partial_T^2 \ln P_0^*(T) = -\theta_T + \sigma^2 T$$

$$\boxed{\theta_T = -\partial_T^2 \ln P_0^*(T) + \sigma^2 T}$$

Can interpolate to find $P_0(T)$ $\forall T$.

$$P_0^*(T) = e^{-y_0(T)\,T}$$

Vasicek Model

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Vasicek Model (Ornstein - Uhlenbeck process)

($\kappa > 0$) mean-reverting level

$$r_n = r_{n-1} + \kappa(\theta - r_{n-1})\Delta t + \sigma\sqrt{\Delta t}\, x_n .$$

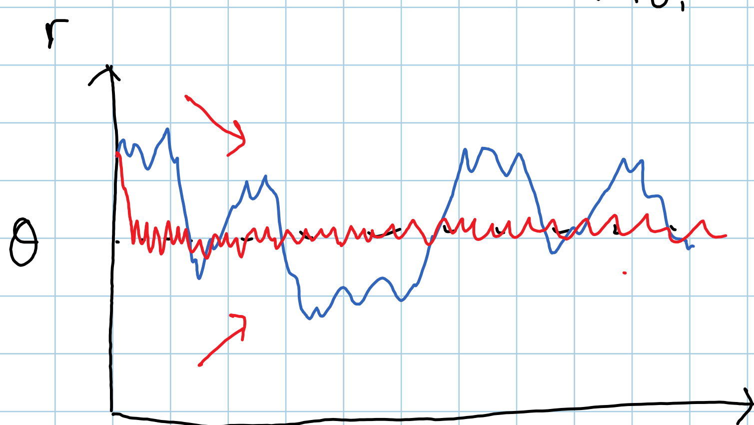
± 1 $q = 1/2$

rate of mean-reversion

volatility.

$r > \theta$, < 0

$r < \theta$, > 0



Mean-reversion.

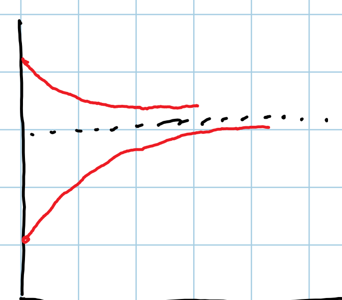
[$\sigma = 0$.

$$r_n = r_{n-1} + \kappa(\theta - r_{n-1})\Delta t$$

$$\frac{r_n - r_{n-1}}{\Delta t} = \kappa(\theta - r_{n-1})$$

$$\dot{r} = \kappa(\theta - r)$$

$$\Rightarrow \quad r = r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t})$$]



$$r_n = r_{n-1} + \kappa(\theta - r_{n-1})\Delta t + \sigma\sqrt{\Delta t}\, x_n$$

$$r_n = r_{n-1} + \kappa(\theta - r_{n-1})\Delta t + \sigma\sqrt{\Delta t}\, x_n$$

$$= \underbrace{\kappa\theta\Delta t}_{a} + \underbrace{(1-\kappa\Delta t)}_{b} r_{n-1} + \sigma\sqrt{\Delta t}\, x_n$$

(Auto-regressive order 1 AR(1) model)

$$= a + b(a + b\, r_{n-2} + \sigma\sqrt{\Delta t}\, x_{n-1}) + \sigma\sqrt{\Delta t}\, x_n$$

$$= a(1+b) + b^2 r_{n-2} + \sigma\sqrt{\Delta t}\,(x_n + b\, x_{n-1})$$

$$= a(1+b) + b^2(a + b\, r_{n-3} + \sigma\sqrt{\Delta t}\, x_{n-2}) + \sigma\sqrt{\Delta t}(x_n + b\, x_{n-1})$$

$$= a(1+b+b^2) + b^3 r_{n-3} + \sigma\sqrt{\Delta t}\,(x_n + b\, x_{n-1} + b^2 x_{n-2})$$

$$= \dots$$

$$= a\sum_{n=1}^{N} b^{n-1} + \underbrace{b^N}_{\to e^{-\kappa T}} r_0 + \overbrace{\sigma\sqrt{\Delta t}\sum_{n=1}^{N} x_{N-n+1}\, b^{n-1}}^{A}$$

$$\hookrightarrow a\,\frac{1-b^N}{1-b} = \frac{1-(1-\kappa\, T/N)^N}{\cancel{(1-(1-\kappa\,\Delta t))}}\cdot \cancel{\kappa\theta\Delta t} \longrightarrow (1-e^{-\kappa T})\,\theta$$

$$\mathbb{E}^{Q}[A] = 0$$

$$\mathbb{V}^{Q}[A] = \sigma^2\,\Delta t \sum_{n=1}^{N} 1\cdot b^{2(n-1)}$$

$$= \sigma^2\,\Delta t\,\frac{1-b^{2N}}{1-b^2} = \sigma^2\,\cancel{\Delta t}\,\frac{1-\overbrace{((1-\kappa\, T/N)^N)^2}^{\to e^{-2\kappa T}}}{1-(1-\kappa\,\Delta t)^2}$$

$$\hookrightarrow 2\kappa \cancel{\Delta t} - \kappa^2 \Delta t^2$$

$$\rightarrow \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa T}\right)$$

$$\boxed{r_T \overset{d}{=} r_0 e^{-\kappa T} + (1 - e^{-\kappa T})\theta + \Big(\underbrace{\frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa T}\right)}_{\xrightarrow[T \to +\infty]{} \frac{\sigma^2}{2\kappa}}\Big)^{1/2} Z}$$

$$Z \underset{Q}{\sim} N(0,1)$$

$$I_T = \int_0^T r_u \, du = \sum_{n=1}^{N} r_{n-1} \Delta t$$

$$r_n = r_{n-1} + \kappa(\theta - r_{n-1})\Delta t + \sigma\sqrt{\Delta t}\, x_n$$

$$\Rightarrow \quad r_n - r_{n-1} = \kappa\theta\Delta t - \kappa r_{n-1}\Delta t + \sigma\sqrt{\Delta t}\, x_n$$

$$r_N - r_0 = \kappa\theta T - \kappa \underbrace{\sum_{n=1}^{N} r_{n-1}\Delta t}_{\hookrightarrow I_T} + \sigma\sqrt{\Delta t}\sum_{n=1}^{N} x_n$$

$$I_T \xrightarrow[N \to \infty]{Q} N\Big(\underbrace{m}_{\hookrightarrow\, r_0 (1 - e^{-\kappa T})/\kappa + \theta g(T)} \,;\, \underbrace{v}_{\hookrightarrow\, h(T)}\Big)$$

$$P_0(T) = \mathbb{E}^{Q}\left[e^{-I_T}\right] = \exp\left\{ - r_0 \frac{1 - e^{-\kappa T}}{\kappa} - \theta g(T)\right.$$

$$+ \frac{1}{2} h(T) \Big\}$$

affine model: Vasicek & Ho-Lee are of this type

IRS and Simulation

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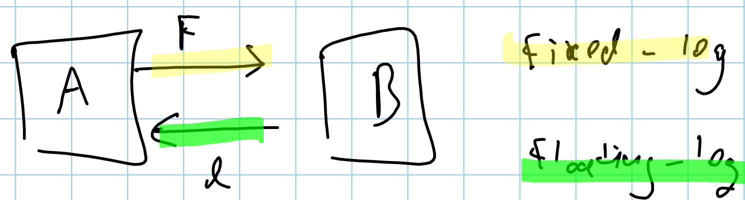
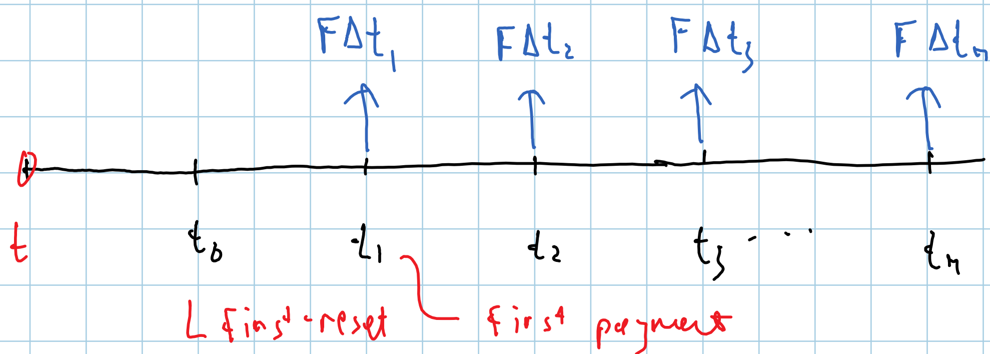
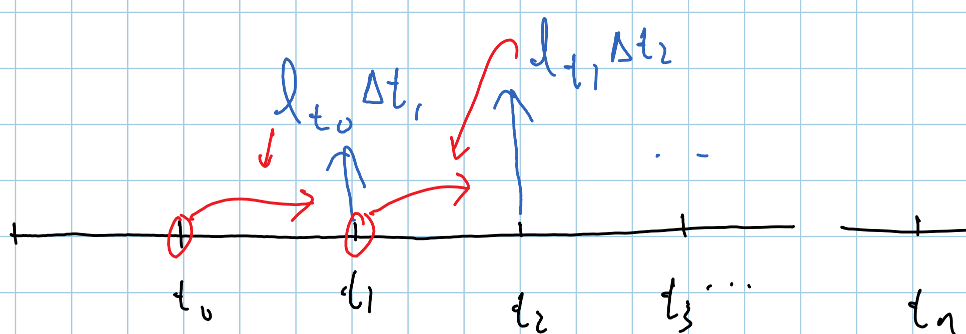
Interest Rate Swaps (IRS)





$$V_t^{fix} = \sum_{m=1}^{n} F \Delta t_m \, P_t(t_m)$$



$$V_t^{fl} = P_t(t_0) - P_t(t_n)$$

$+1$ t_0 - bond

-1 t_n - bond $1 \longrightarrow (1 + l_{t_0} \Delta t_1)$

$\longrightarrow (1 + l_{t_1} \Delta t_2)$ -1

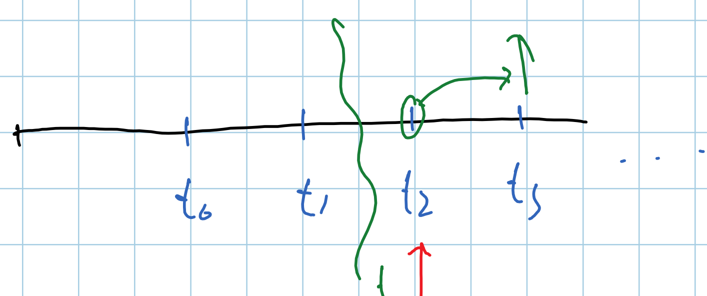
$\longrightarrow (1 + \ell_{t_1} \Delta t_2)$ $\quad -1$

$\hookrightarrow \cdots \hookrightarrow (1 + \ell_{t_{n-1}} \Delta t_n)$

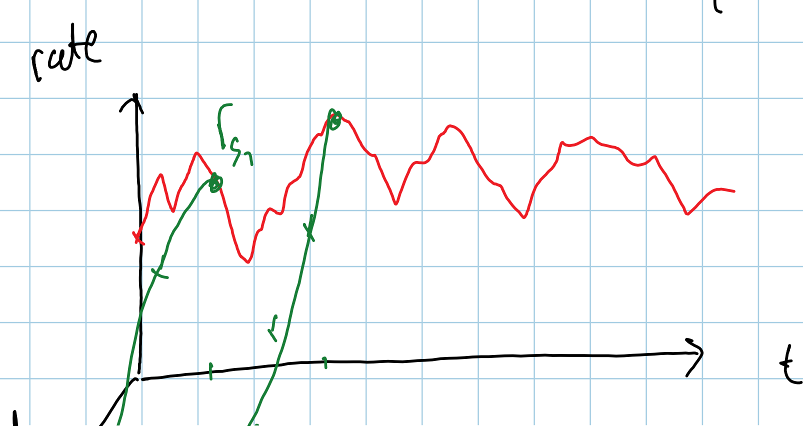
fixed rate F is chosen s.t. $V_0^{fl} = V_0^{fix}$

$$\Rightarrow \boxed{F = \frac{P_0(t_0) - P_0(t_n)}{\underbrace{\sum_{m=1}^{n} \Delta t_m P_0(t_m)}_{\text{annuity}}}}$$

initial swap rate



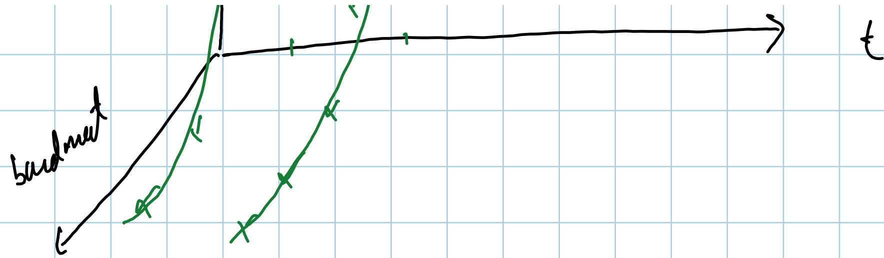
$$S_t = \frac{P_t(t_{k_t}) - P_t(t_n)}{\sum_{m=k_t+1}^{n} \Delta t_m P_t(t_m)}$$



$P_{s_1}(s_1 + n\frac{1}{12}; r_{s_1})$

$\uparrow$

$\rightarrow$ yield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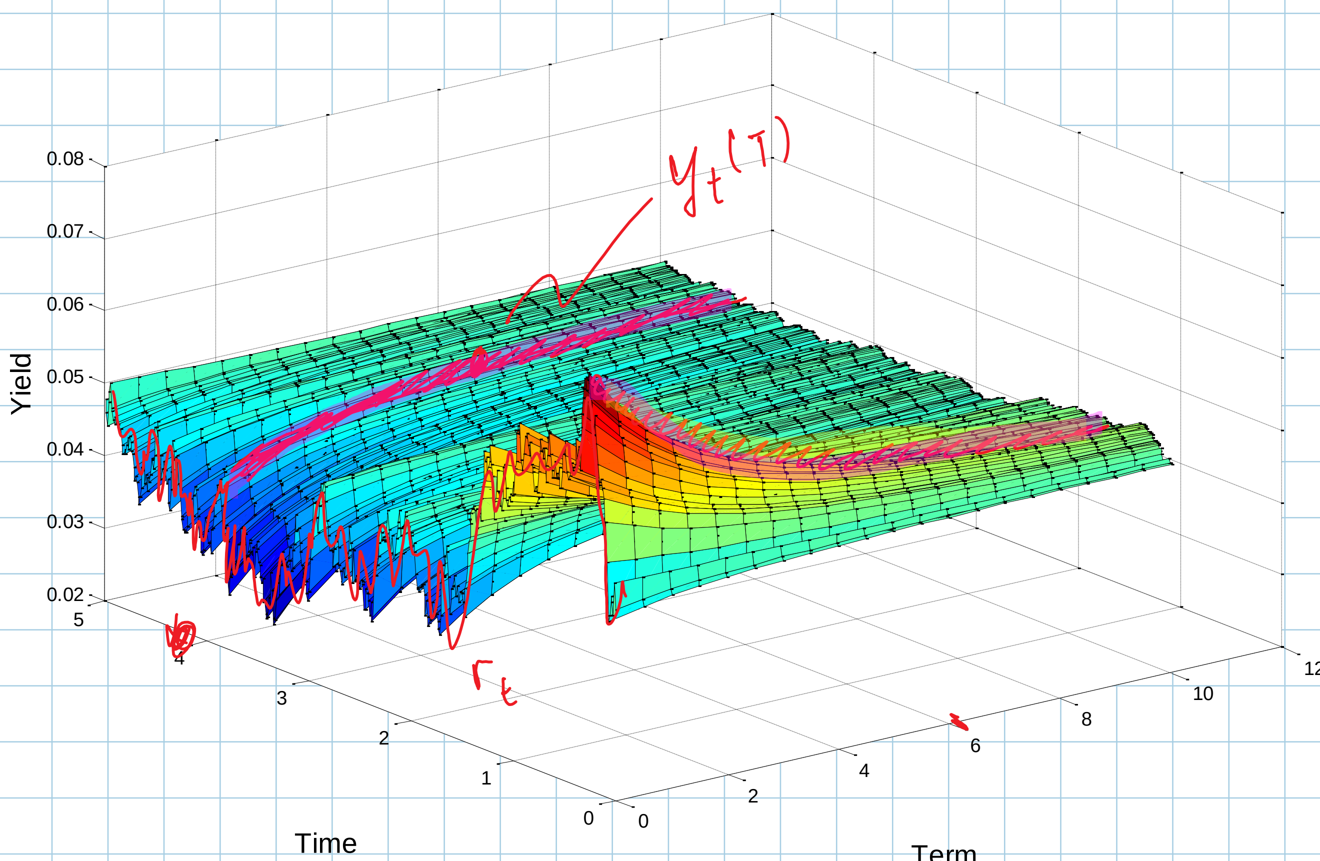
Yield
0.08
0.07
0.06
0.05
0.04
0.03
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Time
5
4
3
2
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Term
0
2
4
6
8
10
12
$y_t(T)$
r_t