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10:33 AM

1d [** 2] $\exp\{a W_T + b Z_T\}$.

NB: $X := a W_T + b Z_T$ is normal distributed

$\mathbb{E}[X] = 0$

$Var[X] = a^2\, Var[W_T] + b^2\, Var[Z_T] + 2ab\, Cov[W_T, Z_T]$

$= (a^2 + b^2 + 2ab\rho)T := \sigma^2 T$

$\Rightarrow \boxed{\mathbb{E}[e^{aW_T + bZ_T}] = e^{\frac{1}{2}\sigma^2 T}}$

$\& \ \mathbb{E}[e^{2(aW_T + bZ_T)}] = e^{\frac{1}{2} 4\sigma^2 T}$

$\therefore \ Var[e^{aW_T + bZ_T}] = e^{\sigma^2 T}(e^{\sigma^2 T} - 1)$

1f (f) [**] $\int_0^T W_s Z_s\, ds$.

NB: $\mathbb{E}[W_s Z_s] = \mathbb{E}[W_s(\rho W_s + \sqrt{1-\rho^2}\, W_s^{\perp})]$

$= \rho s$

$\Rightarrow \mathbb{E}[\int_0^T W_s Z_s\, ds] = \int_0^T \mathbb{E}[W_s Z_s]\, ds$

$= \int_0^T \rho s\, ds = \frac{1}{2} T^2 \rho$

for variance need

$\mathbb{E}[(\int_0^T W_s Z_s\, ds)^2] = \mathbb{E}[\int_0^T \int_0^T W_s Z_s W_t Z_t\, ds\, dt]$

$= \mathbb{E}[2\int_0^T \int_0^t W_s Z_s W_t Z_t\, ds\, dt]$ ← follows from symmetry of integrand

$= 2\int_0^T \int_0^t \mathbb{E}[W_s Z_s W_t Z_t]\, ds\, dt$

now compute expectation (with $s < t$)

$\mathbb{E}[W_s Z_s W_t Z_t]$

$$= \mathbb{E}[\ W_s Z_s\ (W_s + (W_t - W_s))\ (Z_s + (Z_t - Z_s))]$$

$$= \mathbb{E}[\ W_s Z_s\ (W_s Z_s + W_s(Z_t - Z_s) + Z_s(W_t - W_s) + (W_t - W_s)(Z_t - Z_s))]$$

$$= \mathbb{E}[W_s^2 Z_s^2] + \mathbb{E}[W_s^2 Z_s (Z_t - Z_s)]$$

$$+ \mathbb{E}[W_s Z_s^2 (W_t - W_s)] + \mathbb{E}[W_s Z_s (W_t - W_s)(Z_t - Z_s)]$$

$$\text{term 1} = \mathbb{E}[\ W_s^2\ (\rho W_s + \sqrt{1-\rho^2}\ W_s^{\perp})^2]$$

$$= \mathbb{E}[\ W_s^2\ (\rho^2 W_s^2 + 2\rho\sqrt{1-\rho^2}\ W_s W_s^{\perp} + (1-\rho^2)(W_s^{\perp})^2)]$$

$$= \rho^2\ \mathbb{E}[W_s^4] + 2\rho\sqrt{1-\rho^2}\ \mathbb{E}[W_s^3 W_s^{\perp}] + (1-\rho^2)\ \mathbb{E}[W_s^2 (W_s^{\perp})^2]$$

$$= \rho^2\ 3s^2 + 2\rho\sqrt{1-\rho^2}\ \underbrace{\mathbb{E}[W_s^3]}_{0}\underbrace{\mathbb{E}[W_s^{\perp}]}_{0} + (1-\rho^2)\ \mathbb{E}[W_s^2]\mathbb{E}[(W_s^{\perp})^2]$$

$$= s^2\ (3\rho^2 + 1 - \rho^2)$$

$$= s^2\ (1 + 2\rho^2)$$

$$\text{term 2} = \mathbb{E}[W_s^2 Z_s (Z_t - Z_s)] = \mathbb{E}[W_s^2 Z_s]\overbrace{\mathbb{E}[Z_t - Z_s]}^{0} = 0$$

$$\text{term 3} = \mathbb{E}[W_s Z_s^2 (W_t - W_s)] = \mathbb{E}[W_s Z_s^2]\overbrace{\mathbb{E}[W_t - W_s]}^{0} = 0$$

$$\text{term 4} = \mathbb{E}[W_s Z_s (W_t - W_s)(Z_t - Z_s)]$$

$$= \mathbb{E}[W_s Z_s]\mathbb{E}[(W_t - W_s)(Z_t - Z_s)]$$

$$= \rho s\ \rho(t-s) = \rho^2 s(t-s)$$

$$\therefore\ \mathbb{E}[W_s Z_s W_t Z_t] = (1+2\rho^2)s^2 + \rho^2\ s(t-s) \quad (\text{for } s < t)$$

$$\therefore\ \mathbb{E}\left[\left(\int_0^T W_s Z_s\, ds\right)^2\right] = 2\int_0^T\int_0^t \left[(1+2\rho^2)s^2 + \rho^2\ s(t-s)\right] ds\, dt$$

$$= 2\int_0^T\left[(1+2\rho^2)\frac{1}{3}t^3 + \rho^2\left(\frac{1}{2} - \frac{1}{3}\right)t^3\right] dt$$

$$= \frac{2+5\rho^2}{3}\int_0^T t^3\, dt$$

$$= \frac{2+5\rho^2}{12}\ T^4$$

$$\therefore\ \mathrm{Var}\left[\int_0^T W_s Z_s\, ds\right] = \left(\frac{2+5\rho^2}{12} - \frac{1}{4}\rho^2\right)T^4 = \frac{1+\rho^2}{6}\ T^4$$

3g (g) [**] $Y_t = \int_0^t s\, e^{-W_s}\, dW_s$.

$\mathbb{E}[Y_t] = 0$ since it is an Ito integral.

$$\mathbb{E}[Y_t^2] = \mathbb{E}\left[\int_0^t s^2 e^{-2W_s}\, ds\right] \quad \text{by Ito's issometry}$$

$$= \int_0^t s^2\, \mathbb{E}[e^{-2W_s}]\, ds$$

$$= \int_0^t s^2 e^{\frac{1}{2}4s}\, ds$$

$$= \frac{1}{4}\left[(1-2t+2t^2)e^{2t} - 1\right] = Var[Y_t]$$

cosinder $X_t = t e^{-W_t}$, then

$$dX_t = \left(e^{-W_t} + \frac{1}{2} t e^{-W_t}\right) dt - t e^{-W_t} dW_t$$

$$\therefore\ X_t - X_0 = \int_0^t (1 + \tfrac{1}{2}s) e^{-W_s}\, ds - \int_0^t s e^{-W_s}\, dW_s$$

$$\therefore\ \int_0^t s e^{-W_s}\, dW_s = -t e^{-W_t} + \int_0^t (1+\tfrac{1}{2}s) e^{-W_s}\, ds$$

3 (i) [**] $Y_t = \int_0^t s W_s \, dZ_s$.

$\mathbb{E}[Y_t] = 0$ since Y_t is an Ito integral

$$\mathbb{E}[Y_t^2] = \mathbb{E}\left[\int_0^t s^2 W_s^2\, ds\right] \quad \text{by Ito's issometry (see proof below)}$$

$$= \int_0^t s^2\, \mathbb{E}[W_s^2]\, ds$$

$$= \int_0^t s^3\, ds = \frac{1}{4} t^4 = Var[Y_t]$$

consider $X_t = t W_t Z_t$, then

$$dX_t = (W_t Z_t + \rho t)\, dt + t Z_t\, dW_t + t W_t\, dZ_t$$

$$\Rightarrow\ t W_t Z_t = \int_0^t (W_s Z_s + \rho s)\, ds + \int_0^t s Z_s\, dW_s + \int_0^t s W_s\, dZ_s$$

$$\Rightarrow\ \int_0^t s W_s\, dZ_s = t W_t Z_t - \int_0^t (W_s Z_s + \rho s)\, ds - \int_0^t s Z_s\, dW_s$$

Ito's issometry (more general case)

Suppose that

$$Y_t = \int_0^t g_s \, dZ_s$$ where g_s may depend on s, W_s & Z_s,

$$= \lim_{\|\Pi\| \to 0} \sum_{k=1}^{n} \underbrace{g_{t_{k-1}} (Z_{t_k} - Z_{t_{k-1}})}_{A_k}$$

I will show that

$$\mathbb{E}[Y_t] = 0$$

$$\mathbb{E}[Y_t^2] = \mathbb{E}\left[\int_0^t g_s^2 \, ds\right]$$ same result as when g_s is a fn. only of Z_s and s.

consider $\sum_{k=1}^{n} A_k$

let's find mean & variance ...

$$\mathbb{E}[A_k] = \mathbb{E}[g_{t_{k-1}} (Z_{t_k} - Z_{t_{k-1}})]$$

$$= \mathbb{E}[\mathbb{E}[g_{t_{k-1}} (Z_{t_k} - Z_{t_{k-1}}) | \mathcal{F}_{t_{k-1}}]]$$

$$= \mathbb{E}[g_{t_{k-1}} \underbrace{\mathbb{E}[(Z_{t_k} - Z_{t_{k-1}}) | \mathcal{F}_{t_{k-1}}]}_{\to 0}] = 0$$

suppose $k < l$, then

$$\mathbb{E}[A_k A_l]$$

$$= \mathbb{E}[g_{t_{k-1}}(Z_{t_k} - Z_{t_{k-1}})\, g_{t_{l-1}}(Z_{t_l} - Z_{t_{l-1}})]$$

$$= \mathbb{E}[\mathbb{E}[g_{t_{k-1}}(Z_{t_k} - Z_{t_{k-1}})\, g_{t_{l-1}}(Z_{t_l} - Z_{t_{l-1}}) \mid \mathcal{F}_{t_{l-1}}]]$$

$$= \mathbb{E}[g_{t_{k-1}}(Z_{t_k} - Z_{t_{k-1}})\, g_{t_{l-1}} \underbrace{\mathbb{E}[Z_{t_l} - Z_{t_{l-1}} \mid \mathcal{F}_{t_{l-1}}]}_{\to 0}]$$

$$= 0$$

can also argue by saying $Z_{t_l} - Z_{t_{l-1}}$ is independent of $g_{t_{k-1}}(Z_{t_k} - Z_{t_{k-1}})\, g_{t_{l-1}}$.

also,

$$\mathbb{E}[A_k^2] = \mathbb{E}[g^2_{t_{k-1}}(Z_{t_k} - Z_{t_{k-1}})^2]$$

$$= \mathbb{E}[g^2_{t_{k-1}}\, \mathbb{E}[(Z_{t_k} - Z_{t_{k-1}})^2 \mid \mathcal{F}_{t_{k-1}}]]$$

$$= \mathbb{E}[g^2_{t_{k-1}}(t_k - t_{k-1})]$$

$$\therefore \mathbb{E}[\sum_k A_k] = 0 \quad \therefore \mathbb{E}[\lim_{\|\Pi\| \to 0} \sum_k A_k] = 0$$

$$\mathrm{Var}[\sum_k A_k] = \mathbb{E}[(\sum_k A_k)^2]$$

$$= \mathbb{E}[\sum_k A_k^2] + 2\,\mathbb{E}[\sum_{k<l} A_k A_l]$$

$$= \sum_k \mathbb{E}[g^2_{t_{k-1}}(t_k - t_{k-1})] + 0$$

$$= \mathbb{E}[\sum_k g^2_{t_{k-1}}(t_k - t_{k-1})]$$

$$\xrightarrow[\|\Pi\| \to 0]{} \mathbb{E}[\int_0^t g_s^2\, ds]$$

5 a) iv. [**] $d((S_t)^\alpha (U_t)^\beta)$ for $\alpha \neq 0; \beta \neq 0$.

$$X_t := S_t^\alpha U_t^\beta$$

$$\begin{aligned} dX_t = \Big\{ & 0 + \alpha S_t^{\alpha-1} U_t^\beta \cdot c_t S_t + \tfrac{1}{2}\alpha(\alpha-1) S_t^{\alpha-2} U_t^\beta \cdot d_t^2 S_t^2 \\ & + S_t^\alpha \beta U_t^{\beta-1} \cdot a_t U_t + \tfrac{1}{2} S_t \beta(\beta-1) U_t^{\beta-2} \cdot b_t^2 U_t^2 \\ & + \rho \alpha S_t^{\alpha-1} \beta U_t^{\beta-1} \cdot b_t U_t \cdot d_t S_t \Big\} dt \\ & + \alpha S_t^{\alpha-1} U_t^\beta \cdot d_t S_t \, dW_t^s \\ & + S_t \beta U_t^{\beta-1} \cdot b_t U_t \, dW_t^u \end{aligned}$$

$$\Rightarrow d(S_t^\alpha U_t^\beta) = S_t^\alpha U_t^\beta \left(\alpha c_t + \tfrac{1}{2}\alpha(\alpha-1) d_t^2 + \beta a_t + \tfrac{1}{2}\beta(\beta-1) b_t^2 + \rho\alpha\beta b_t d_t \right) dt$$
$$+ S_t^\alpha U_t^\beta \left(\alpha d_t \, dW_t^s + \beta b_t \, dW_t^u \right)$$

(b) [**2] Solve the system of SDEs (1).

since they are only coupled through correlation, can solve each individually..

$$\frac{dU_t}{U_t} = a_t \, dt + b_t \, dW_t^u$$

$$\Rightarrow d(\ln U_t) = \left(a_t - \tfrac{1}{2} b_t^2 \right) dt + b_t \, dW_t^u$$

$$\Rightarrow \ln U_t - \ln U_0 = \int_0^t \left(a_s - \tfrac{1}{2} b_s^2\right) ds + \int_0^t b_s \, dW_s^u$$

$$\Rightarrow U_t = U_0 \exp\left\{ \int_0^t \left(a_s - \tfrac{1}{2} b_s^2 \right) ds + \int_0^t b_s \, dW_s^u \right\}$$

similarly,

$$S_t = S_0 \exp\left\{ \int_0^t \left(c_s - \tfrac{1}{2} d_s^2 \right) ds + \int_0^t d_s \, dW_s^s \right\}$$

4a (a) $\int W_s^2 dW_s = \frac{1}{3} W_t^3 - \int_0^t W_s \, ds$

$$\text{Consider } A \triangleq \int_0^t W_s^2 \, dW_s - \tfrac{1}{3} W_t^3 + \int_0^t W_s \, ds$$

$$A = \lim_{\|\Pi\| \downarrow 0} \sum_k \left\{ W_{t_{k-1}}^2 \Delta W_k - \tfrac{1}{3} \left(W_{t_k}^3 - W_{t_{k-1}}^3 \right) + W_{t_{k-1}} \Delta t_k \right\}$$

$$= \lim_{\|\Pi\| \downarrow 0} \sum_k \left\{ \tfrac{1}{3} \underbrace{\left(W_{t_k} - W_{t_{k-1}} \right)^3}_{B_k} \right.$$

$$- \underbrace{W_{t_{k-1}} \left((W_{t_k} - W_{t_{k-1}})^2 - \Delta t_k \right)}_{\hookrightarrow C_k} \}$$

Notice:

① $$\mathbb{E}\left[\sum_k B_k\right] = \sum_k \mathbb{E}[B_k] = 0$$

$$\mathbb{V}\left[\sum_k B_k\right] = \sum_k \mathbb{V}[B_k] = \sum_k \mathbb{E}[\Delta W_k^6]$$

$$= \sum_k (\Delta t_k)^3 \, \mathbb{E}[Z^6]$$

$\hookrightarrow$ std. normal

$\mathbb{E}[Z^6] = 15 < +\infty$

$$\leq 15 \, \|\Pi\|^2 \sum_k \Delta t_k = 15 \, \|\Pi\|^2 \, t \longrightarrow 0$$

$$\therefore \sum_k B_k \longrightarrow 0 \quad \text{a.s.}$$

② $$\mathbb{E}\left[\sum_k C_k\right] = \sum_k \mathbb{E}[C_k]$$

$$= \sum_k \underbrace{\mathbb{E}[W_{t_{k-1}}]}_{= t_{k-1}} \underbrace{\mathbb{E}[(W_{t_k} - W_{t_{k-1}})^2 - (t_k - t_{k-1})]}_{= 0} = 0$$

$$\mathbb{V}\left[\sum_k C_k\right] = 2 \sum_{k<l} \mathbb{C}[C_k, C_l] + \sum_k \mathbb{V}[C_k]$$

independent

$$= 2 \sum_{k<l} \mathbb{E}\left[W_{t_{k-1}} W_{t_{l-1}} (\Delta W_k^2 - \Delta t_k)(\Delta W_l^2 - \Delta t_l)\right]$$

$$+ \sum_k \mathbb{E}\left[W_{t_{k-1}}^2 (\Delta W_k^2 - \Delta t_k)^2\right]$$

independent

$$= \sum_k \mathbb{E}[W_{t_{k-1}}^2] \, \mathbb{E}[(\Delta W_k^2 - \Delta t_k)^2]$$

$$= \sum_k t_{k-1} \times \left[3(\Delta t_k)^2 - 2(\Delta t_k)^2 + (\Delta t_k)^2\right]$$

$$= 2 \sum_k t_{k-1} (\Delta t_k)^2$$

—

$$\leq 2\,\|\pi\| \underbrace{\sum_k t_{k-1}\,\Delta t_k}_{\xrightarrow[\|\pi\|\downarrow 0]{} \frac{1}{2}t^2} \longrightarrow 0$$

$$\therefore \lim_{\|\pi\|\downarrow 0} \sum_k A_k = 0 \quad a.s. \quad \Rightarrow \quad A = 0 \quad a.s.$$

$$\Rightarrow \int_0^t W_s^2\,dW_s - \frac{1}{3}W_t^3 + \int_0^t W_s\,ds = 0 \quad a.s.$$

$$\therefore \int_0^t W_s^2\,dW_s = \frac{1}{3}W_t^3 - \int_0^t W_s\,ds \quad a.s.$$

4 5) Show $\int_0^t W_s\,dZ_s + \int_0^t Z_s\,dW_s = W_t Z_t - \rho t$

consider $A \triangleq \int_0^t W_s\,dZ_s + \int_0^t Z_s\,dW_s - W_t Z_t + \rho t$

$$A = \lim_{\|\pi\|\downarrow 0} \sum_k A_k$$

$$A_k \triangleq W_{t_{k-1}}\,\Delta Z_k + Z_{t_{k-1}}\,\Delta W_k - (W_{t_k} Z_{t_k} - W_{t_{k-1}} Z_{t_{k-1}}) + \rho\,\Delta t_k$$

$$= -\Delta W_k\,\Delta Z_k + \rho\,\Delta t_k$$

Note: $\mathbb{E}[\sum_k A_k] = \sum_k \mathbb{E}[A_k] = 0$

$$\mathbb{V}[\sum_k A_k] = \sum_k \mathbb{V}[A_k] = \sum_k \mathbb{V}[\Delta W_k\,\Delta Z_k]$$

$$= \sum_k \mathbb{E}[(\Delta W_k)^2(\Delta Z_k)^2]$$

$$= \sum_k \mathbb{E}[(N_1\sqrt{\Delta t_k})^2(\rho N_1\sqrt{\Delta t_k} + \sqrt{1-\rho^2}\,N_2\sqrt{\Delta t_k})^2]$$

$$= \sum_k (\Delta t_k)^2 \rho$$

$$\leq \rho\,\|\pi\| \sum_k \Delta t_k = \rho\,\|\pi\|\,t$$

$$\longrightarrow 0$$

$\|\pi\| \to 0$

$\therefore \quad \sum_k A_k \to 0 \text{ a.s.} \qquad \therefore \quad A = 0 \text{ a.s.}$

$\therefore \quad \int_0^t W_s \, dZ_s + \int_0^t Z_s \, dW_s = W_t Z_t - \rho t \quad \text{a.s.}$