An Introduction to Algorithmic Trading
Research in Options 2013

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many thanks to

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Introduction

Introduction

Aim is to learn about

- **Algorithmic Trading (AT)**: The use of computer algorithms that make trading decisions, submit orders, and manage those orders after submission.
    - For example, the SEC reports that in the NYSE between 2005 and 2009: consolidated average daily share volume increased 181%; average speed of execution for small, immediately executable (marketable) orders shrunk from 10.1 to 0.7 seconds; consolidated average daily trades increased 662%; and consolidated average trade size decreased from 724 to 268 shares, SEC (2010).
- We will learn about
    - VWAP and TWAP
    - Optimal liquidation
    - Pairs trading

Introduction

- Why AT? One example
  - Institutional or large players need to trade (buy and sell) large amounts of securities. These quantities are too large for the market to process without prices moving in the 'wrong direction' (slippage).
  - Thus, large orders are broken up in small ones and these are traded over time (minutes, hours, days, weeks, or even months) and across different venues.
  - Deciding how to break up and execute a large order can mean saving millions of dollars for large players

Introduction

Aim is to learn about

- **High Frequency (HF) Trading**: refers to the subset of AT trading strategies that are characterised by their reliance on speed differences relative to other traders to make profits based on short-term predictions and also by the objective to hold essentially no inventories for more than a very short period of time.
- We will learn about
  - Market making
  - Momentum and short-term-alpha strategies
  - Risk metrics for HFT

Exchanges

Exchanges

An exchange is a ‘place’ where ‘people’ meet to buy/sell securities: shares, commodities, derivatives, etc

- Order Driven Market:
  - All buyers and sellers display the prices and quantities at which they wish to buy or sell a particular security, This is the opposite of a quote driven market, which is one that only displays bids and asks of designated market makers and specialists for a specific security.
- Quote Driven Market:
  - Designated market makers and specialists display bids and asks for a specific security.

Order Driven Market

- All participants can post **limit buy or sell orders**
- limit orders show an intention to buy or sell and must indicate the amount of shares and price at which the agent is willing to trade
  - limit buy order with the highest price is known as the **best bid**
  - limit sell order with the lowest price is known as the **best offer**
  - The best bid/ask is also called the **touch**
- The difference between the best bid and offer is called the **spread**

Evolution of markets

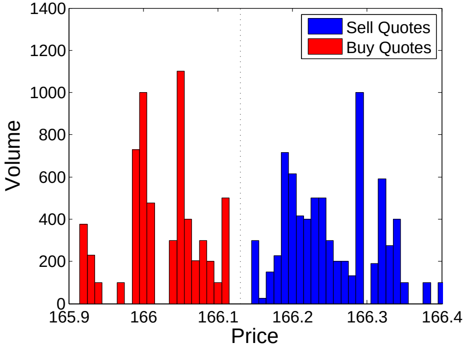
- Old days brokerage model: Ring a broker, broker sends order to the pit and after screaming and hand signalling the order is executed.
- Electronic market: Ring or use internet to contact broker who sends the order to the electronic exchange (no screaming)
- Direct Access Market: clients send orders directly to market

But to which market? ARCA-NYSE: electronic platform of NYSE, BATS (Kansas), BEX: Boston Equity Exchange, CBSX (CBOE Stock Exchange), CSXZ (Chicago Stock Exchange), DRCTEDGE (Direct Edge, Jersey City, NJ), ISE (International Securities Exchange), ISLAND (Acquired by Nasdaq in 2003), LAVA (Citigroup), NSX (National Stock Exchange, Chicago) TRACKECN (Track ECN), ChiX, LSE (London Stock Exchange), etc, etc

Limit Order Book

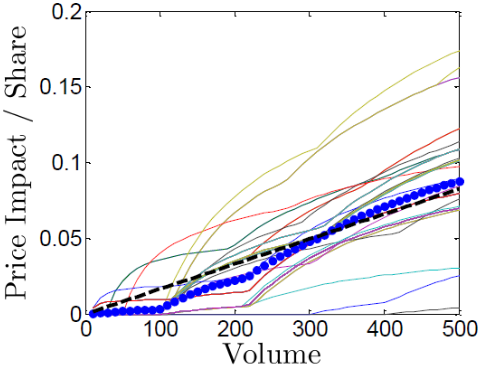
Limit Order Book

- **Limit orders** are accumulated in the limit order book (**LOB**) until they find a counterparty for execution or are cancelled
- The counterparty is a **market order** which is an order to buy or sell an amount of shares, regardless of the price, and is **immediately executed** against the **best prices**



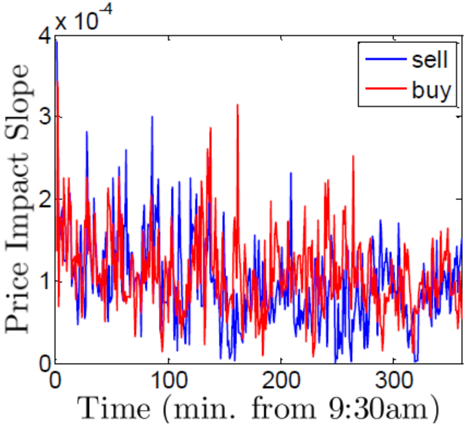
Limit Order Book

Immediate Execution Costs



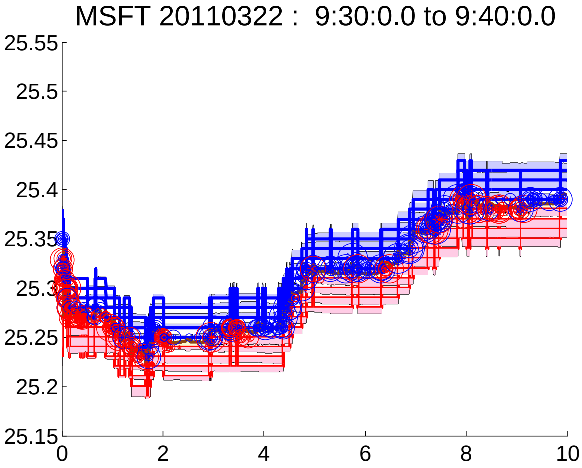
Limit Order Book

Impact dynamics throughout the day



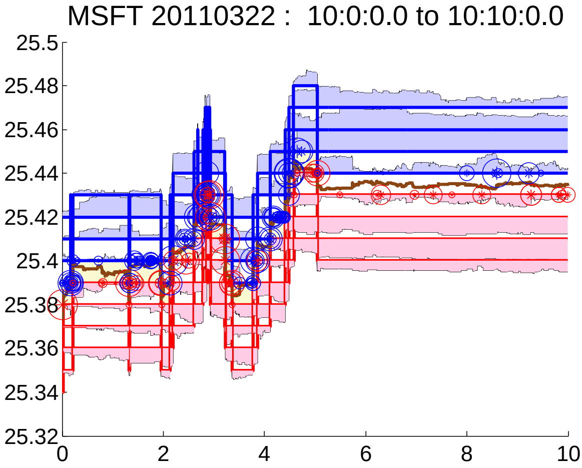
Limit Order Book

Trade Activity



Limit Order Book

Trade Activity



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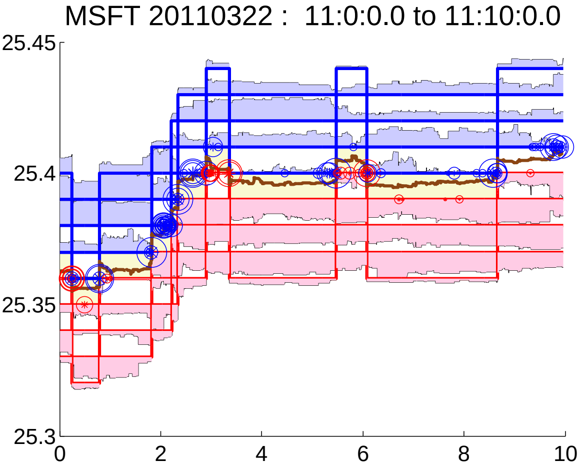

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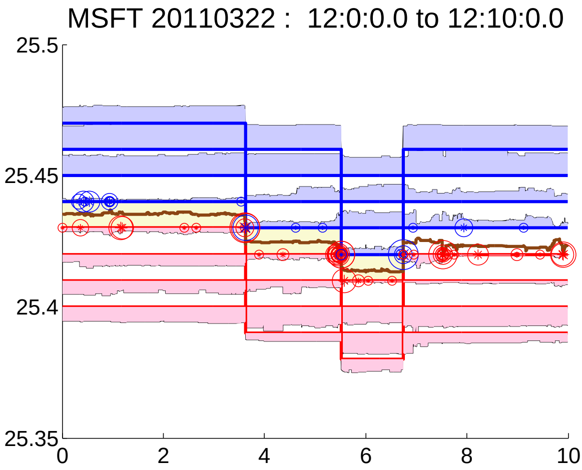

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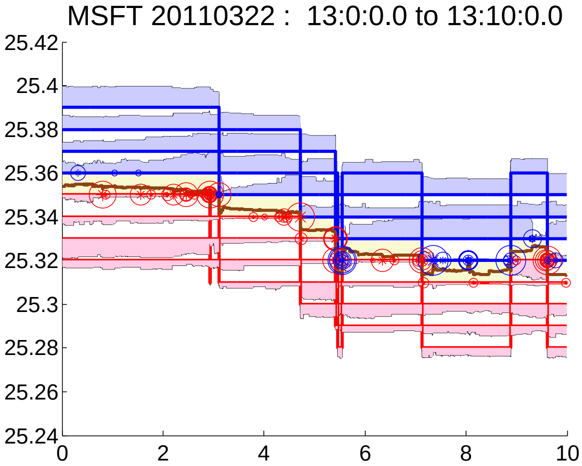

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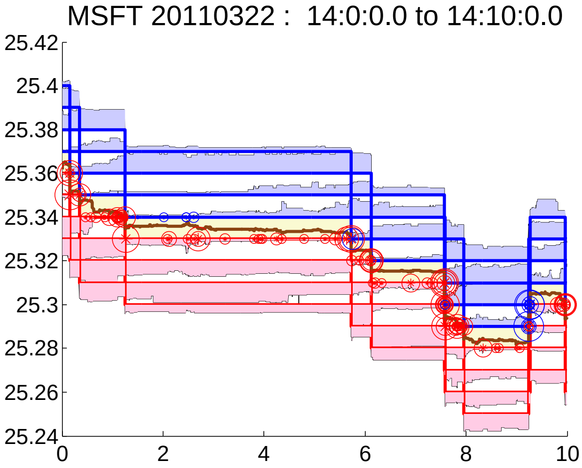

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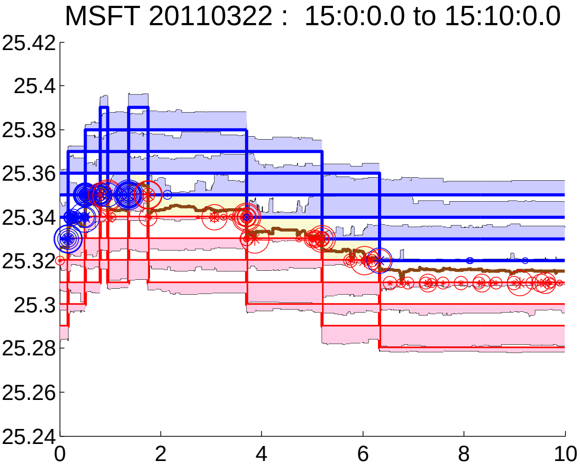

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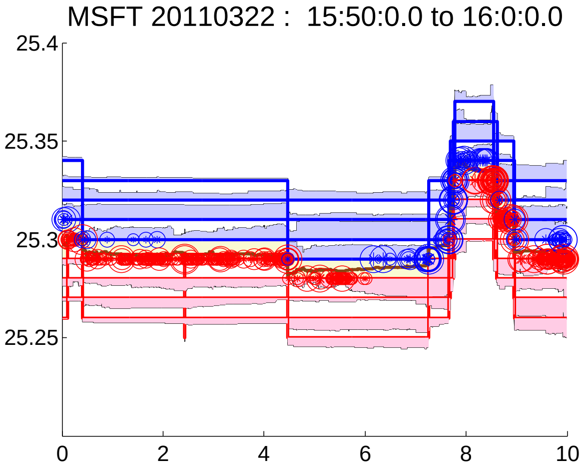


Order Imbalance

Order Imbalance

- Order Imbalance $\rho \in [0, 1]$

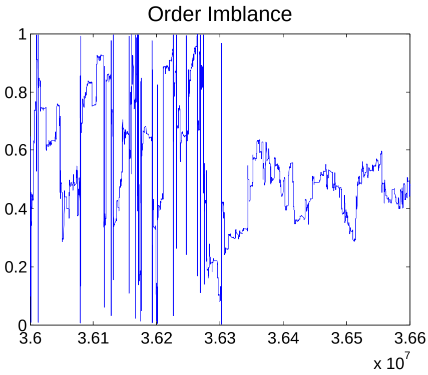
$$\rho_t = \frac{V_t^b}{V_t^a + V_t^b}$$

- It is a good predictor of trade direction (ORIT June 21, 2011)

ρ	# Buy Orders	# Sell Orders
All	**756 (67%)**	396 (33 %)
> 0.5	**568 (79%)**	155 (21%)
> 0.75	**320 (84%)**	60 (16%)
< 0.5	168 (43%)	**225 (57%)**
< 0.25	39 (25%)	**116 (75%)**

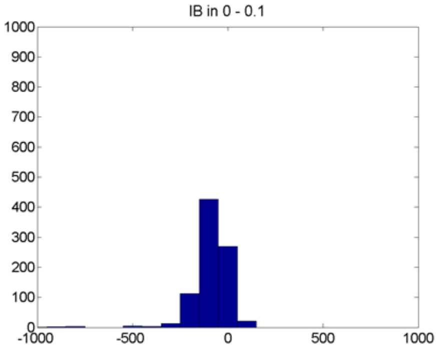
Order Imbalance

A slice of imbalance for MSFT 10:00am to 10:10am on 22 Mar 2011



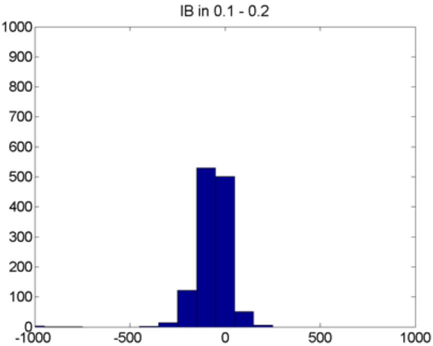
Order Imbalance

Midprice change pre/post MO event with $\rho \in [0, 0.1)$



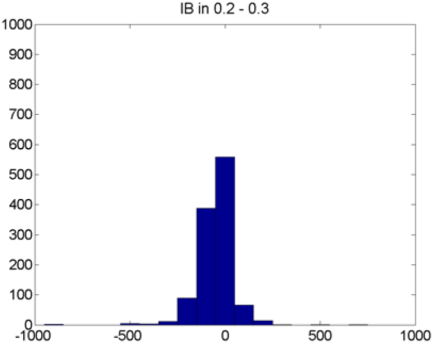
Order Imbalance

Midprice change pre/post MO event with $\rho \in [0.1, 0.2)$



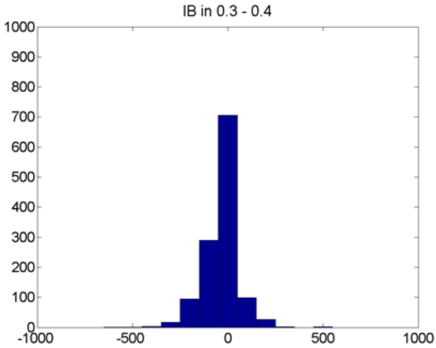
Order Imbalance

Midprice change pre/post MO event with $\rho \in [0.2, 0.3)$



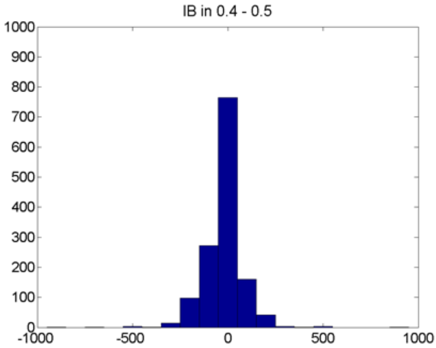
Order Imbalance

Midprice change pre/post MO event with $\rho \in [0.3, 0.4)$



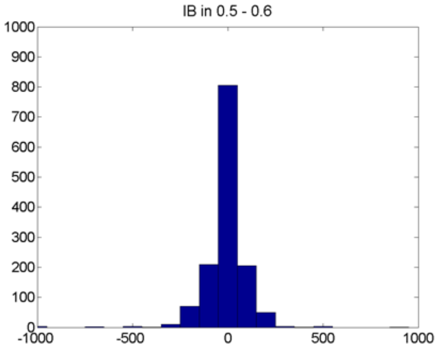
Order Imbalance

Midprice change pre/post MO event with $\rho \in [0.4, 0.5)$



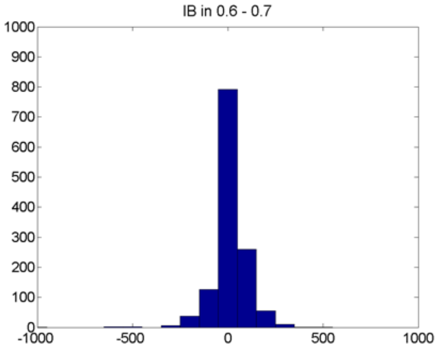
Order Imbalance

Midprice change pre/post MO event with $\rho \in [0.5, 0.6)$



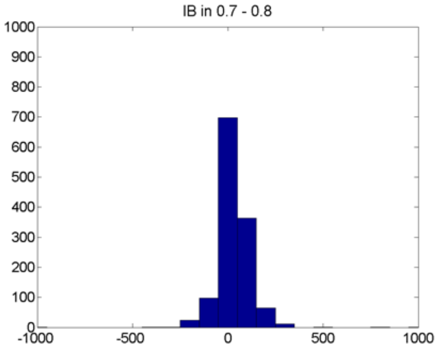
Order Imbalance

Midprice change pre/post MO event with $\rho \in [0.6, 0.7)$



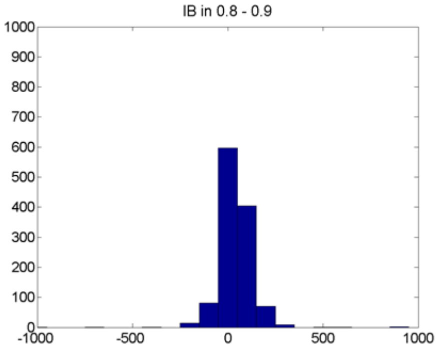
Order Imbalance

Midprice change pre/post MO event with $\rho \in [0.7, 0.8)$



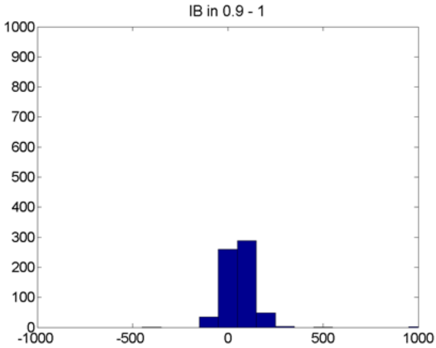
Order Imbalance

Midprice change pre/post MO event with $\rho \in [0.8, 0.9)$



Order Imbalance

Midprice change pre/post MO event with $\rho \in [0.9, 1.0]$



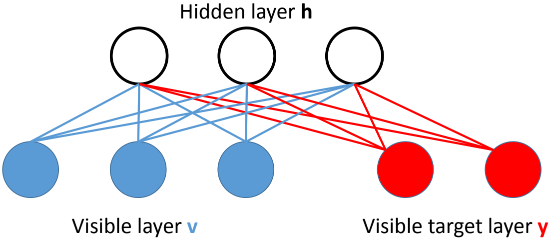
Statistical Machine Learning Models

Statistical Machine Learning

- Statistical machine learning models are quite varied
  - Supervise learning (inputs and outputs known – i.e. data is "labeled")
    - Support vector machines
    - Gaussian random fields
    - Restricted Boltzman Machines
  - Unsupervise learning (only inputs known – no "labels")
    - Clustering (k-means, mixture models)
    - Hidden Markov Models
    - Blind signal separation (PCA, SVD, independent component analysis)

Restricted Boltzman Machines

- An **RBM** is a probabilistic model with a **hidden** and a **visible** layer with connections only between layers:



- Joint distribution of observing a configuration is

$$p(y, v, h \,|\, \theta) = \frac{1}{Z(\theta)}\, e^{-E(y,v,h|\theta)}$$

where the "Energy" of the configuration is

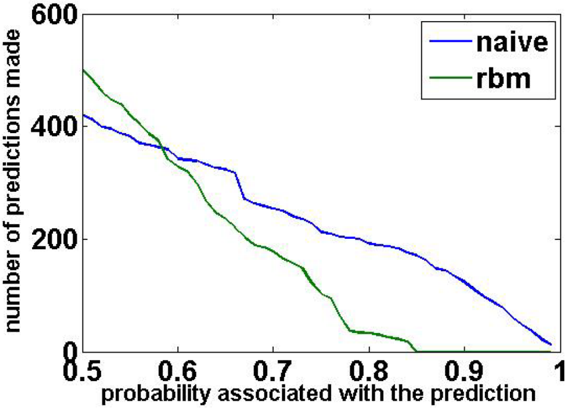
$$E = -\sum_{r=1}^{R}\sum_{k=1}^{K} v_r\,\omega_{rk}\,h_k - \sum_{i=1}^{C}\sum_{k=1}^{K} y_i\,\beta_{ik}\,h_k - \sum_{r=1}^{R} b_r\,v_r - \sum_{k=1}^{K} c_k\,h_k - \sum_{i=1}^{C} d_i\,y_i$$

Restricted Boltzman Machines

- Training of the model (i.e., learning ω, β, b, c and d)
  - compute all binary states

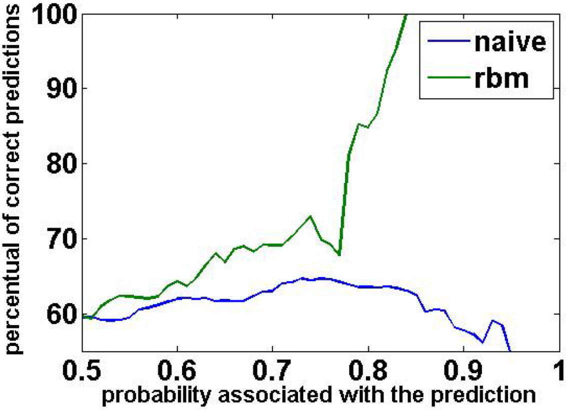
Restricted Boltzman Machines

The number of predictions using the given confidence cutoff



Restricted Boltzman Machines

The % correct predictions using the given confidence cutoff



Who are you trading with? Is she better informed?

Imagine that you are a market maker... (part I)

Glosten-Milgrom model

- An asset is worth either $\overline{V}$ or $\underline{V}$ with prob $\frac{1}{2}$
- MM sets the bid and ask prices
- A trader arrives and buys OR sells the asset to the MM
- The trader is informed with probability α
- **Informed traders** know the asset outcome before trading
- **Uniformed traders** buy/sell with probability $\frac{1}{2}$

Imagine that you are a market maker... (part II)

Glosten-Milgrom model

- What are the rational prices that a risk-neutral MM sets?
  - Naive answer is simply to set

$$ask = bid = \mathbb{E}[\ V\ ]$$

  - The MM will then be **adversely selected**...
    - When trading with uninformed trader... no losses on average.
    - When they sell... they sometimes sell to an informed trader who knows the price was going up
    - When they buy... they sometimes buy from an informed trader who knows the price was going down

Imagine that you are a market maker... (part III)

Glosten-Milgrom model

- To account for the potential of being adversely selected, the MM sets instead

$$ask = \mathbb{E}[\ V \mid \text{MO is buy}\ ]$$
$$bid = \mathbb{E}[\ V \mid \text{MO is sell}\ ]$$

- After some simple computations one finds

$$ask = \mathbb{E}[\ V\ ] + \frac{\alpha}{2}\,(\overline{V} - \underline{V})$$
$$bid = \mathbb{E}[\ V\ ] - \frac{\alpha}{2}\,(\overline{V} - \underline{V})$$

- Therefore

$$\boxed{spread = \alpha\,(\overline{V} - \underline{V})}$$

Optimal Liquidation

Optimal Liquidation

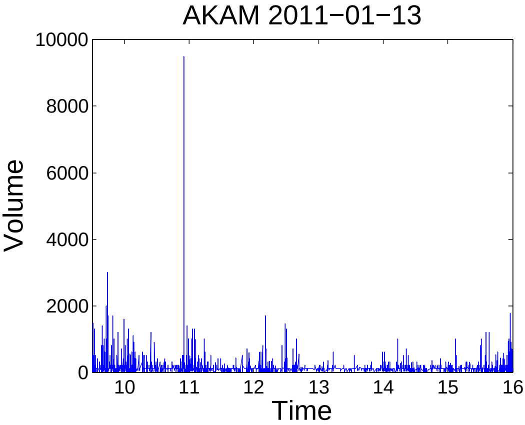
- Imagine you have a **large position** in an asset to **liquidate**
- If you **sell all at once**, then your orders will walk the book and receive you a **large market impact**
- If you **sell slowly**, then you have **large uncertainty** in the price
- You need to **tradeoff impact and uncertainty**

Optimal Liquidation

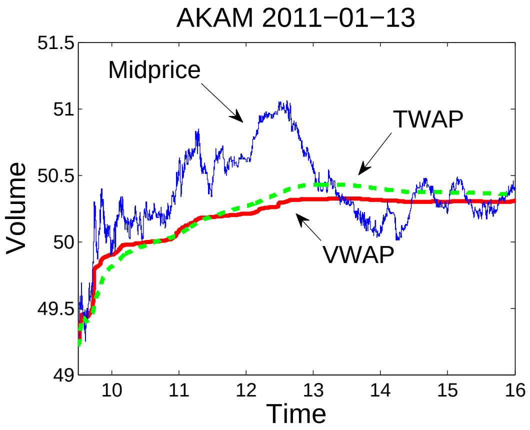
Several **target prices** are used in practice

- **VWAP** (volume weighted average price)
  - The objective is to obtain the volume-weighted average price (might be a self-fulfilling prophecy)
  - Must predict the volume
- **TWAP** (time weighted average price)
  - The objective is to obtain the time-weighted average price
- **POV** (percentage of volume)
  - Use a constant participation rate, say β
  - For example, if the agent still needs to purchase Q shares, the algo computes the volume traded over a time window, say V, and then executes $\min(Q, \beta V)$

Optimal Liquidation



Optimal Liquidation



Optimal Liquidation

Almgren-Chriss framework minimizes **arrival price slippage**

- Trader **sells at a rate** of ν_t
- Without trading, **fundamental price** is a **Brownian motion**
- But **trading permanently impacts prices**

$$dS_t = -\mathbf{b}\,\nu_\mathbf{t}\,dt + \sigma\,dW_t$$

- Also, trader does not receive S_t when purchasing, instead there is a **temporary impact** and the trader receives

$$S_t^* = S_t - \mathbf{a}\,\nu_\mathbf{t}$$

per share

Optimal Liquidation

Almgren-Chriss framework minimizes **arrival price slippage**

- **Trader's wealth** X from liquidating shares is therefore

$$X_t = \int_0^t (S_u - a\,\nu_u)\,\nu\,du$$

- The trader's **optimization problem** is to **maximize expected profits** but **penalize inventories**

$$\sup_{\nu \in \mathcal{A}} \mathbb{E}\left[X_T - \phi\sigma^2 \int_0^T q_u^2\,du\right]$$

where **trader's inventory** remaining is

$$q_t = Q - \int_0^t \nu_u\,du$$

Optimal Liquidation

Almgren-Chriss framework minimizes **arrival price slippage**

- This is a **stochastic control problem**
- In general solved via the **dynamic programming principle** (DPP) leading to **dynamic programming equations** (DPEs)
- Basic ideas
  - Introduce a time indexed collection of **performance criteria** for an **arbitrary strategy** (not necessarily optimal)

$$H^{(\nu)}(t,x,S,q) = \mathbb{E}_{t,x,S,q}\left[X_T^{(\nu)} - \phi\sigma^2 \int_t^T \left(q_u^{(\nu)}\right)^2 du\right]$$

  - Use iterated expectations to show that

$$H^{(\nu)}(t,x,S,q) = \mathbb{E}_{t,x,S,q}\left[H^{(\nu)}(\tau, S_\tau^\nu, q_\tau^\nu) - \phi\sigma^2 \int_t^\tau \left(q_u^{(\nu)}\right)^2 du\right]$$

Optimal Liquidation

Almgren-Chriss framework minimizes **arrival price slippage**

- Choose the arbitrary strategy

$$\nu_t = \begin{cases} \nu_t, & t \in [t, \tau] \quad \text{arbitrary} \\ \nu_t^*. & t \in (\tau, T] \quad \text{optimal} \end{cases}$$

- Let $H(t, S, q)$ denote the optimal strategy, then

$$H^{(\nu)}(t, x, S, q) \leq \mathbb{E}_{t,x,S,q}\left[H(\tau, S_\tau^{\nu^*}, q_\tau^{\nu}) - \phi\sigma^2 \int_t^\tau \left(q_u^{(\nu)}\right)^2 du\right]$$

- Equality holds for the optimal strategy

$$H(t, x, S, q) = \mathbb{E}_{t,x,S,q}\left[H(\tau, S_\tau^{\nu^*}, q_\tau^{\nu^*}) - \phi\sigma^2 \int_t^\tau \left(q_u^{(\nu^*)}\right)^2 du\right]$$

Optimal Liquidation

Almgren-Chriss framework minimizes **arrival price slippage**

- Write the above in infinitesimal form using Ito's lemma to find

$$0 = \partial_t H_{+} \tfrac{1}{2}\sigma^2 \partial_{SS} \\ + \sup_{\nu} \Big\{ (S - a\,\nu)\,\nu\,\partial_x H + (S - b\,\nu)\,\partial_S H - \nu\,\partial_q H \Big\}$$

- optimal **trading speed** in feedback form is

$$\nu^* = \frac{1}{2\,a\,\partial_x H}\,(S\partial_x + b\,\partial_S + \partial_q)H$$

Optimal Liquidation

Almgren-Chriss framework minimizes **arrival price slippage**

- Substitute back into the DPE to find a **non-linear PDE** for H

$$0 = \left(\partial_t + \frac{1}{2}\sigma^2\partial_{SS}\right) H - \phi\, q^2 + \frac{1}{4a\partial_x H}\left[\left(S\partial_x + b\partial_S + \partial_q\right) H\right]^2$$

- An ansatz $H = x + S\,q + h(t)\,q^2$ in fact reduces the problem to a **Riccati ODE**

$$0 = \partial_t h - \phi + \tfrac{1}{4a}(b - 2\,h)^2$$

Optimal Liquidation

Almgren-Chriss framework minimizes **arrival price slippage**

- Substitute everything back, one then finds

$$\nu^* = \frac{1}{a} h_t \, q_t$$

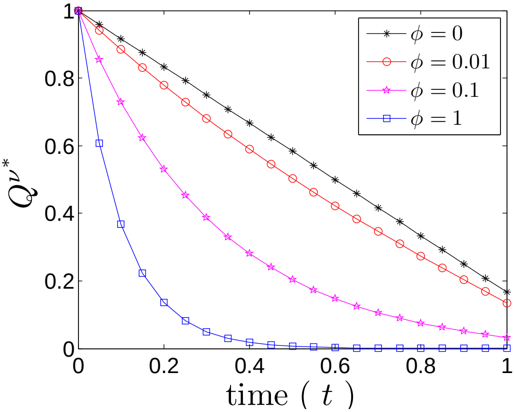
  It is deterministic function scaled by current remaining inventory

- More explicitly, one can show that

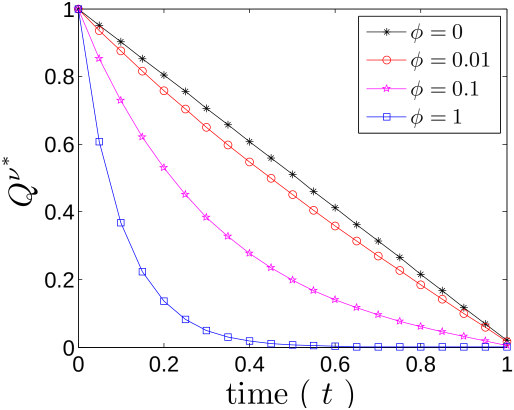
$$q_t = \frac{\sinh(\gamma(T-t))}{\sinh(\gamma T)}$$

  where $\gamma = \sqrt{\phi/a}$

Optimal Liquidation



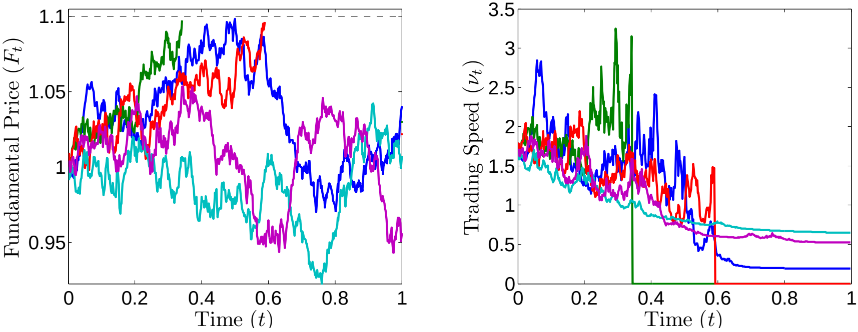
Optimal Liquidation



optimal strategy is **independent** of the **asset price**!

Optimal Acquisition

If there is a **limit price**, i.e., trader must acquire all shares prior to an upper bound... see J. & Kinzebulatov (2013)



Now **optimal strategy** is **dependent** on **asset price**

Optimal Acquisition

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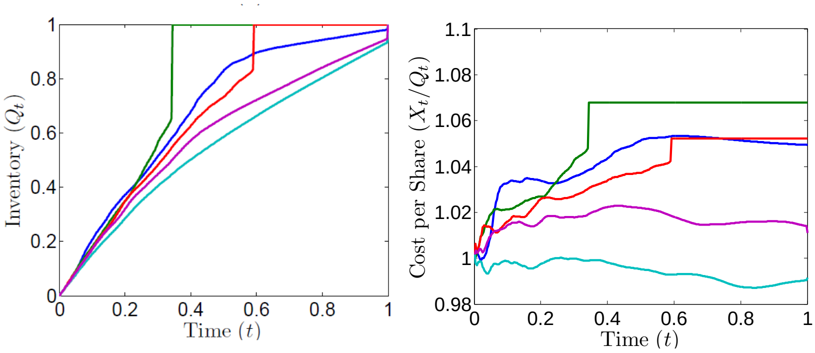


Now **optimal strategy** is **dependent** on **asset price**

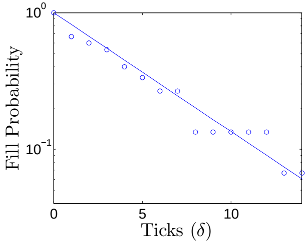
Market Making

Market Making

- The **market maker's problem** is to **find prices** at which to post **limit buy/sell order** to **profit** from round-trip trades
- The **benchmark models**: Ho & Stoll (81), Avellanda & Stoikov (08) J. & Cartea (12)
- Need to account for
  - market order **arrival rate**
  - **Probability** that you are **filled** at a given level
  - **midprice dynamics**

Market Making

- **Market orders.** arrive at jump times of a **Poisson process** $M_t^{\pm}$ with **intensity** $\lambda^{\pm}$
- **Fill probabilities.** MOs fill posted LOs with $p = e^{-\kappa^{\pm}\delta^{\pm}}$



- **Midprice.** The midprice is a drifted Brownian motion

$$dS_t = \mu\, dt + \sigma\, dW_t .$$

Market Making

- Other ingredients to solve MMs profit maximisation problem
  - $N_t^{\pm}$ – **counting process** for **filled limit orders**

$$N_t = \int_0^t \int_{c^{\pm}\delta_s^{\pm}}^{+\infty} \mu^{\pm}(dy, ds)$$

$\mu^{\pm}(dy, ds)$ is a **Poisson Random Measure** with mean-measure

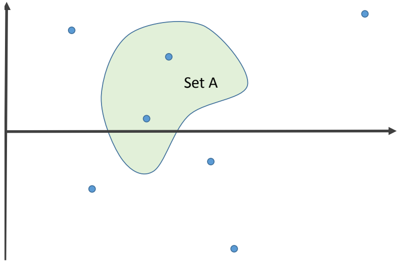
$$\nu^{\pm}(dy, ds) = \lambda^{\pm} A^{\pm} e^{-A^{\pm} y} \, dy \, ds$$

  - Total market orders are given in terms of the random measure

$$M_t^{\pm} = \int_0^t \int_0^{+\infty} \mu^{\pm}(dy, ds)$$

Market Making

A **PRM** μ with **mean measure** ν is a **random measure** on a measurable space $(E, \mathcal{E})$



such that

- $\mu(A)$ is **Poisson random variable**, with mean $\nu(A)$, $\forall\ A \in \mathcal{E}$
- $\mu(A_1)$, $\mu(A_2)$, $\ldots$ are **independent** for countable disjoint $A_1, A_2, \cdots \in \mathcal{E}$

Market Making

- Other ingredients to solve MMs profit maximisation problem
  - q_t – **agent's inventory**

$$q_t = \mathbf{N_t^-} - \mathbf{N_t^+}$$

  - X_t – **agent's cash process**

$$dX_t = \underbrace{(S_t + \delta_t^+)\, d\mathbf{N_t^+}}_{\text{gains from filled sells}} - \underbrace{(S_t - \delta_t^-)\, d\mathbf{N_t^-}}_{\text{costs from filled buys}}$$

- Choose $\delta^{\pm}$ to **maximise expected penalised wealth**:

$$H(t,x,q,S) = \sup_{\delta_t^{\pm} \in \mathcal{A}} \mathbb{E}\left[X_T + q_T(S_T - \ell(q_T)) \middle| \mathcal{F}_t \right]$$

Market Making

- A **DPP** shows that H is the unique viscosity solution to

$$\partial_t H + \underbrace{\left(\mu\partial_S + \tfrac{1}{2}\sigma^2\partial_{SS}\right) H}_{\text{mid-price Diffusion}}$$

$$+ \sup_{\delta^+}\left\{\lambda^+ \underbrace{e^{-\kappa^+\delta^+}}_{\text{prob. filled sell LO}} \underbrace{\Big(H(t, x + (S + \delta^+), q - 1, S) - H(t, x, q, S)\Big)}_{\text{change due to filled sell LO}}\right\}$$

$$+ \sup_{\delta^-}\left\{\lambda^- \underbrace{e^{-\kappa^-\delta^-}}_{\text{prob. filled buy LO}} \underbrace{\Big(H(t, x - (S - \delta^-), q + 1, S) - H(t, x, q, S)\Big)}_{\text{change due to filled buy LO}}\right\} = 0\,,$$

s.t. $H(T, x, q, S) = x + q(S - \ell(q))\,.$

Optimal Sell Spreads – No Ambiguity

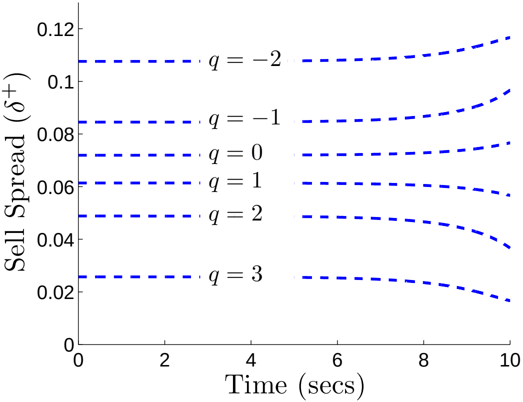
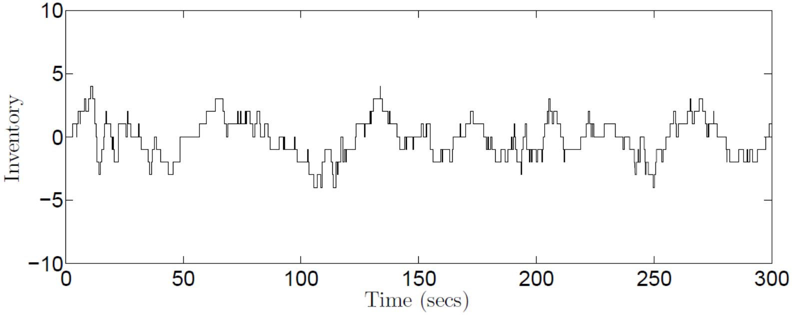


Figure : Optimal sell spreads. Parameter values are $\kappa^{\pm} = 15$, $\lambda^{\pm} = 2$, $\sigma = 0.01$, $\mu = 0$, $\ell(q) = \alpha q$, $\alpha = 0.01$, $\overline{q} = -\underline{q} = 3$ and $T = 10$ seconds.

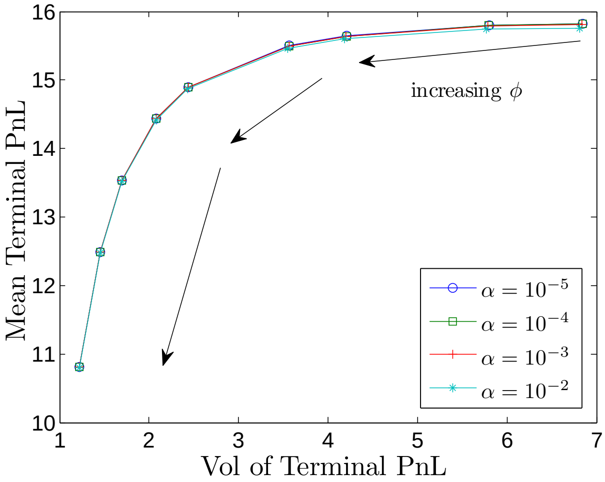
Market Making

Sample inventory path...



Market Making

Risk-Reward profile...



Informed Trading

Informed Trading

- Suppose an agent makes a **prediction** $\widehat{S}_T$ about S_T
  - E.g., IBM will go up within two hours by $2
- **Naive strategy** is to execute a market order to
  - buy asset if $\widehat{S}_T > S_t + \Delta/2$
  - sell asset if $\widehat{S}_T < S_t - \Delta/2$
- How can one...
  - incorporate **prediction uncertainty**?
  - include both **limit and market orders**?
  - **learn** from the realised dynamics of
    - the asset midprice?
    - other assets midprice?

Informed Trading

- Asset midprice S is a **randomised Brownian bridge** (rBB)

$$S_t = S_0 + \sigma\, \boldsymbol{\beta}_{t,T} + \frac{t}{T}\, \mathbf{D}$$

$\boldsymbol{\beta}_{t,T}$ – standard **Brownian bridge** representing "noise" fluctuations in the LOB due to the action of traders

$\mathbf{D}$ – the **random change** in asset price

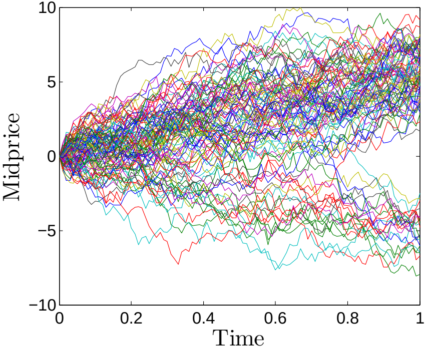
- The midprice history $(S_u)_{u \le t}$ generates the filtration $\mathcal{F}_t$
  - cannot distinguish signal (D) from noise ($\beta_{t,T}$)
- Borrows ideas from Brody, Hughston, Macrina (08) on information based asset pricing
  - But differs since in BHM: S_t generates filtration, but the asset price $X_t = \mathbb{E}[D|\mathcal{F}_t]$ and is a martingale

Informed Trading

- In algo trading problems, many model S_t as Brownian motion
- When $D|_{\mathcal{F}_0} \sim \mathcal{N}(0, \sigma^2 T)$, **rBB reduces** to **Brownian motion**
- This new approach allows the agent to include their future views on the asset price directly in the assumed dynamics, e.g.,
  - **Uninformed traders** have prior $D|_{\mathcal{F}_0} \sim \mathcal{N}(0, 1)$
  - **Informed traders** have prior $D|_{\mathcal{F}_0} \sim \mathcal{N}(a, b^2)$
    obtained from internal models – e.g. by using LOB shape, order flow, statistical machine learning, etc...

Informed Trading

A prediction of upward/downward movement within bands...



Informed Trading

- All traders observe S_t and learn about D, e.g.,
  - **Uninformed traders** have prior $D|_{\mathcal{F}_0} \sim \mathcal{N}(0,1)$
  - **Informed traders** have prior $D|_{\mathcal{F}_0} \sim \mathcal{N}(a,b^2)$
    obtained from internal models – e.g. by using LOB shape, order flow, statistical machine learning, etc...
- The posterior probability conditional on $\mathcal{F}_t$ is given by

$$\mathbb{P}(D \in \mathcal{D} | S_t = s) = \frac{\int_{\mathcal{D}} \exp\left\{x \, \frac{s-s_0}{\sigma^2(T-t)} - x^2 \, \frac{t}{2\sigma^2 T(T-t)}\right\} \; \mu_D(dx)}{\int \exp\left\{x \, \frac{s-s_0}{\sigma^2(T-t)} - x^2 \, \frac{t}{2\sigma^2 T(T-t)}\right\} \; \mu_D(dx)}$$

- Go to slides from other talk...