STA 3000, Fall 2008 — Assignment #3

Due December 4, at start of lecture. Worth 8% of the course grade.

This assignment is to be done by each student individually. You may discuss it in general terms with other students, but the work you hand in should be your own. In particular, you should not leave any discussion of this assignment with any written notes or other recordings, nor receive any written or other material from anyone (except your instructor) by other means such as email.

For all questions, show both the final answer and how you obtained it.

Question 1: Given $\theta \in (0, \infty)$, data $X \in (0, \infty)$ has the $\text{Exp}(1/\theta)$ distribution, for which the density function is $f(x) = (1/\theta)\exp(-x/\theta)$, $E(X) = \theta$ and $\text{Var}(X) = \theta^2$. We use an inverse gamma prior for θ, with paramters $a \in (1, \infty)$ and $b \in (0, \infty)$, for which the density function is $f(\theta) = (b^a/\Gamma(a))\theta^{-a-1}\exp(-b/\theta)$, $E(\theta) = b/(a-1)$ (infinite if $a \leq 1$, but these values are not in the parameter space), and $\text{Var}(\theta) = b^2/(((a-1)^2(a-2)$ (infinite if $a \leq 2$).

a) Find the posterior density for θ given $X = x$.

b) Prove that the formal Bayes rule for estimating θ given data x with squared error loss is $\delta_{a,b}(x) = (x+b)/a$, and find the posterior expected loss when using $\delta_{a,b}$.

c) Find the risk function, $R(\theta, \delta_{a,b})$, of this formal Bayes rule, for all a and b.

d) Find the Bayes risk by integrating $R(\theta, \delta_{a,b})$ with respect to the inverse gamma prior for θ with parameters a and b.

e) Find the risk function of $\delta_1(x) = x$, which is the limit of $\delta_{a,b}$ as $a \to 1$ and $b \to 0$.

f) Prove that δ_1 is inadmissible, and find another decision rule that dominates it.

Question 2: Given $\theta \in R$, data $X_1, \ldots, X_n$ are IID distributed according to the Exp(1) distribution shifted right by θ — ie, we can write $X_i = Z_i + \theta$ with $Z_i \sim \text{Exp}(1)$. We wish to estimate θ, with squared-error loss. One possible estimator is $\delta_0(x) = \overline{x} - 1$, which is unbiased, and equivariant.

1. Find the risk function for δ_0.
2. Find the minimal sufficient statistic for this model, and then by applying the Rao-Blackwell theorem, find an estimator, δ_1, that should have risk at least as small as δ_0.
3. Find the risk function for δ_1.
4. Find the Pitman estimator, δ_2, of θ starting with δ_0. Comment on its relationship with δ_1.