Study Questions #3 for STA 3000

These are for study only, not to hand in for credit

Q1: Suppose that given λ, $X_1, \ldots, X_n$ are IID Poisson(λ). Let the prior for λ be Exp(1).

a) Find a simple expression for the posterior distribution of λ.

b) Find a formal Bayes rule for estimating λ from $X = (X_1, \ldots, X_n)$ with squared error loss, and find the posterior risk, $r(\delta|x)$, for this rule.

c) Find the risk function for the formal Bayes rule you found in (b).

Q2: Consider a decision problem with data $X = (X_1, X_2, X_3)$, in which X_1, X_2, and X_3 are independent given the value of a parameter $\Theta \in \{0, 1, 2, 3\}$, with

$$X_i|\Theta = \theta \;\sim\; \text{Poisson}(1 + I_{\{i\}}(\theta))$$

In other words, if $\Theta = 0$, all X_i are i.i.d. from the Poisson(1) distribution. Otherwise, X_θ is from the Poisson(2) distribution and the other X_i are from the Poisson(1) distribution. After observing X, we take an action $a \in \{0, 1, 2, 3\}$, and suffer loss given by

$$L(\theta, a) \;=\; \begin{cases} 0 & \text{if } a = \theta \\ 1 & \text{if } a = 0 \text{ and } \theta \neq 0 \\ 2 & \text{if } a \neq 0 \text{ and } \theta \neq a \end{cases}$$

Suppose our prior distribution for Θ is

$$P(\Theta = 0) \;=\; 1/2, \qquad P(\Theta = 1) \;=\; P(\Theta = 2) \;=\; P(\Theta = 3) \;=\; 1/6$$

Find the formal Bayes rule for this decision problem. Say explicitly what decision should be taken according to this rule for each of the following data sets, (X_1, X_2, X_3):

$$(4, 0, 0), \quad (5, 0, 1), \quad (0, 5, 6), \quad (0, 4, 5), \quad (5, 6, 5)$$

Q3: Given $\lambda \in (0, \infty)$, X_1 and X_2 are IID with Poisson(λ) distribution. Consider the following two decision rules for estimating λ with squared error loss, based on $X = (X_1, X_2)$:

$$\delta_1(x) \;=\; \frac{x_1 + x_2}{2}$$

$$\delta_2(x) \;=\; \begin{cases} 1 & \text{if } x_1 = 0 \text{ or } x_2 = 0 \\ 2 & \text{otherwise} \end{cases}$$

1. Show that δ_2 does not dominate δ_1.
2. Show that δ_1 does not dominate δ_2.
3. Use the Rao-Blackwell theorem to find a decision rule that dominates δ_2. Give an explicit formula for this decision rule, and find a value of λ for which it has lower risk than δ_2.