# STA 3000, Fall 2009 — Assignment #4

*Due 15 January 2010. Worth 8% of the course grade.*

*This assignment is to be done by each student individually. You may discuss it in general terms with other students, but the work you hand in should be your own. In particular, you should not leave any discussion of this assignment with any written notes or other recordings, nor receive any written or other material from anyone else by other means such as email.*

Consider the problem of estimating $\theta$ from two data points, $X_1,\, X_2 \overset{\text{IID}}{\sim} U(\theta, \theta+1)$. We will look at the following estimator:

$$\delta_0(x) \;=\; x_1 \;-\; 1/2$$

Consider first squared-error loss, for which $L(\theta, a) = (\theta - a)^2$, with $a$ the real-valued estimate.

a) Find the risk function for $\delta_0$.

b) Apply the Rao-Blackwell theorem to find an estimator $\delta_1$ that should have risk at least as small as $\delta_0$. As the sufficient statistic, use the order statistics, $X_{(1)}$, $X_{(2)}$

c) Find the risk function for $\delta_1$.

d) Show that $\delta_1$ is also the Pitman estimator for this problem, by correcting $\delta_0(x)$ by subtracting $E_{\theta=0}[\delta_0(X)|Y = x_1 - x_2]$, where $Y = X_1 - X_2$. Confirm that this is also what you get by finding the mean of the normalized likelihood function.

The Rao-Blackwell theorem applies only to convex loss functions. Consider instead the class, $\mathcal{M}$, of loss functions of the form $L(\theta, a) = f(|\theta - a|)$, with $f$ being a monotonically non-decreasing function (ie, $f(d) \le f(d')$ if $d \le d'$).

e) For the specific case of this model, with $\delta_0$ and $\delta_1$ above, prove that for any loss function in the class $\mathcal{M}$, the risk for $\delta_1$ is at least as small as the risk for $\delta_0$.

Suppose that the model is instead that $X_1$ and $X_2$ are IID from a mixture distribution, with probability $9/10$ that $X_i$ is exactly $\theta + 1/2$ and probability $1/10$ that $X_i$ is drawn from the $U(\theta, \theta+1)$ distribution. In other words, $X_1,\, X_2 \overset{\text{IID}}{\sim} (9/10)\delta_{\theta+1/2} \;+\; (1/10)U(\theta, \theta+1)$, where $\delta_w$ is a point mass at $w$. We will look at the same estimator, $\delta_0$, as above. If we use the order statistics as the sufficient statistic, the Rao-Blackwell theorem applied to $\delta_0$ will give $\delta_1$ as before.

f) Find a loss function in the class $\mathcal{M}$ for which the estimator $\delta_1$ does not have risk at least as small as $\delta_0$.