Study Questions #3 for STA 3000

These are for study only, not to hand in for credit

Q1: Suppose that given λ, $X_1, \ldots, X_n$ are IID Poisson(λ). Let the prior for λ be Exp(1).

a) Find a simple expression for the posterior distribution of λ.

b) Find a formal Bayes rule for estimating λ from $X = (X_1, \ldots, X_n)$ with squared error loss, and find the posterior risk, $r(\delta|x)$, for this rule.

c) Find the risk function for the formal Bayes rule you found in (b).

Q2: Consider a decision problem with data $X = (X_1, X_2, X_3)$, in which X_1, X_2, and X_3 are independent given the value of a parameter $\Theta \in \{0, 1, 2, 3\}$, with

$$X_i|\Theta = \theta \;\sim\; \text{Poisson}(1 + I(i = \theta))$$

In other words, if $\Theta = 0$, all X_i are i.i.d. from the Poisson(1) distribution. Otherwise, X_θ is from the Poisson(2) distribution and the other X_i are from the Poisson(1) distribution. After observing X, we take an action $a \in \{0, 1, 2, 3\}$, and suffer loss given by

$$L(\theta, a) \;=\; \begin{cases} 0 & \text{if } a = \theta \\ 1 & \text{if } a = 0 \text{ and } \theta \neq 0 \\ 2 & \text{if } a \neq 0 \text{ and } \theta \neq a \end{cases}$$

Suppose our prior distribution for Θ is

$$P(\Theta = 0) \;=\; 1/2, \qquad P(\Theta = 1) \;=\; P(\Theta = 2) \;=\; P(\Theta = 3) \;=\; 1/6$$

Find the formal Bayes rule for this decision problem. Say explicitly what decision should be taken according to this rule for each of the following data sets, (X_1, X_2, X_3):

$$(4,\,0,\,0), \quad (5,\,0,\,1), \quad (0,\,5,\,6), \quad (0,\,4,\,5), \quad (5,\,6,\,5)$$

Q3: Given $\theta \in (0, \infty)$, data $X \in (0, \infty)$ has the Exp($1/\theta$) distribution, for which the density function is $f(x) = (1/\theta)\exp(-x/\theta)$, $E(X) = \theta$ and $\text{Var}(X) = \theta^2$. We use an inverse gamma prior for θ, with paramters $a \in (1, \infty)$ and $b \in (0, \infty)$, for which the density function is $f(\theta) = (b^a/\Gamma(a))\theta^{-a-1}\exp(-b/\theta)$, $E(\theta) = b/(a-1)$ (infinite if $a \leq 1$, but these values are not in the parameter space), and $\text{Var}(\theta) = b^2/(((a-1)^2(a-2)$ (infinite if $a \leq 2$).

a) Find the posterior density for θ given $X = x$.

b) Prove that the formal Bayes rule for estimating θ given data x with squared error loss is $\delta_{a,b}(x) = (x + b)/a$, and find the posterior expected loss when using $\delta_{a,b}$.

c) Find the risk function, $R(\theta, \delta_{a,b})$, of this formal Bayes rule, for all a and b.

d) Find the Bayes risk by integrating $R(\theta, \delta_{a,b})$ with respect to the inverse gamma prior for θ with parameters a and b.

e) Find the risk function of $\delta_1(x) = x$, which is the limit of $\delta_{a,b}$ as $a \to 1$ and $b \to 0$.

f) Prove that δ_1 is inadmissible, and find another decision rule that dominates it.

Q4: Suppose a positive integer X is observed from a distribution of the following form:

$$P_\theta(X = x) \;=\; \begin{cases} 3/4 & \text{if } x = \theta \\ 1/4 & \text{if } x = \theta + 1 \\ 0 & \text{otherwise} \end{cases}$$

where the parameter θ is a positive integer (ie, $\Omega = \{1, 2, 3, \ldots\}$). We wish to find a decision rule that estimates θ with $0-1$ loss — ie,

$$L(\theta, \delta(x)) \;=\; \begin{cases} 0 & \text{if } \delta(x) = \theta \\ 1 & \text{if } \delta(x) \neq \theta \end{cases}$$

(1) Find the risk functions for each of the estimators $\delta_1(x) = x$, $\delta_2(x) = x + 1$, and $\delta_3(x) = x - 1$.

(2) Find a Bayes rule for each of the following prior distributions:

$$\begin{aligned} &\text{(a)}\ P(\Theta = \theta) \;=\; 2^{-\theta} \\ &\text{(b)}\ P(\Theta = \theta) \;=\; 3 \cdot 4^{-\theta} \\ &\text{(c)}\ P(\Theta = \theta) \;=\; 2 \cdot 3^{-\theta} \end{aligned}$$

Determine whether or not each of these rules is admissible.

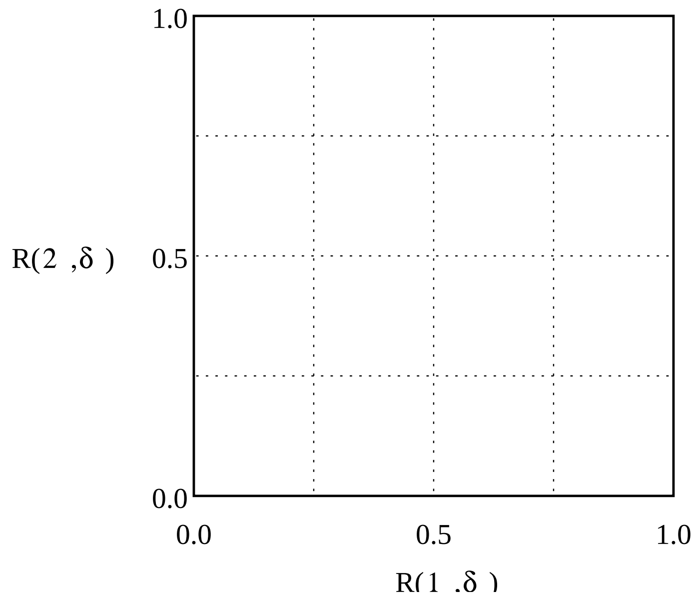
(3) Which of the rules $\delta_1(x) = x$, $\delta_2(x) = x + 1$, and $\delta_3(x) = x - 1$ are admissible?

(4) Find a minimax rule for this decision problem.

Q5: Consider a decision problem in which the data space is $\mathcal{X} = \{0, 1, 2\}$, the action space is $\aleph = \{1, 2\}$, the parameter space is $\Omega = \{1, 2\}$, the loss function is $L(\theta, a) = |\theta - a|$, and the distribution of $X \in \mathcal{X}$ given $\theta \in \Omega$ is $X|\theta \sim \text{binomial}(\theta, 1/2)$.

a) There are eight possible non-randomized decision rules. Find the risk functions for all of them. (Use a tabular presentation, labeling the rules as $\delta_1, \ldots, \delta_8$.)

b) Draw the risk set for this decision problem.



c) Which of these non-randomized decision rules are admissible? For each non-admissible rule, give a rule that dominates it.

d) Which of these non-randomized decision rules are Bayes rules for some prior? For each rule that is Bayes rule, give a prior for which it is a Bayes rule.

e) Find a minimax decision rule (which might be randomized). Is it unique?

f) Is the minimax decision rule you found for (d) a Bayes rule for some prior, and if so, what prior? Is it admissible?

Q6: Consider a decision problem with data space $\mathcal{X} = \{1, 2\}$, parameter space $\Omega = \{\theta_0, \theta_1\}$, and action space $\aleph = \{a_0, a_1\}$. Suppose that the loss function is

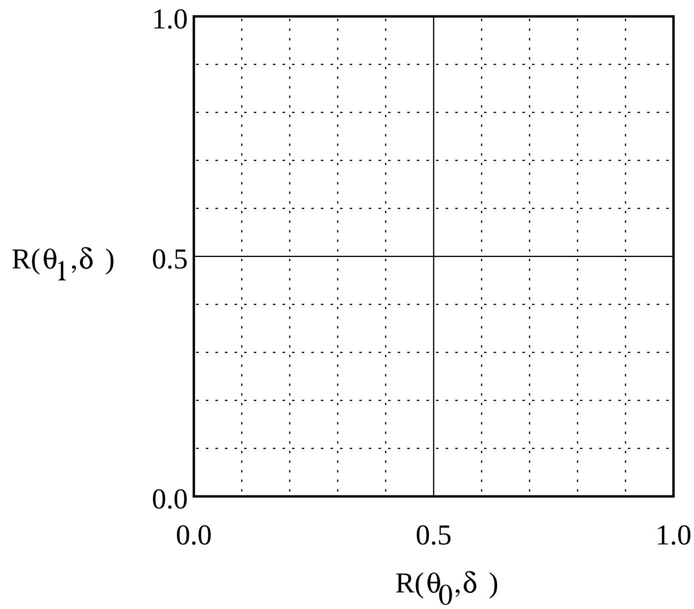
$$L(\theta, a) = \begin{cases} 0.2 & \text{if } \theta = \theta_0 \text{ and } a = a_0 \\ 0.8 & \text{if } \theta = \theta_0 \text{ and } a = a_1 \\ 0.7 & \text{if } \theta = \theta_1 \text{ and } a = a_0 \\ 0.4 & \text{if } \theta = \theta_1 \text{ and } a = a_1 \end{cases}$$

and that the data distributions are given by

$$P_{\theta_0}(x) = \begin{cases} 0.5 & \text{if } x = 1 \\ 0.5 & \text{if } x = 2 \end{cases}$$

$$P_{\theta_1}(x) = \begin{cases} 0.0 & \text{if } x = 1 \\ 1.0 & \text{if } x = 2 \end{cases}$$

1. Draw the risk set for this decision problem in the diagram below:



2. Find a minimax rule for this decision problem. (Write an explicit formula for the decision rule.) Is this the unique minimax rule? Is it admissible? Explain your answers (geometric explanations are OK).

3. Find a Bayes rule for this decision problem when the prior gives equal probability to θ_0 and θ_1. (Write an explicit formula for the decision rule.) Is this Bayes rule unique? Is it admissible? Explain.

4. Are all Bayes rules for this decision problem (for all possible priors) admissible? Explain.