Study Questions #4 for STA 3000

These are for study only, not to hand in for credit

Q1: Given $\lambda \in (0, \infty)$, X_1 and X_2 are IID with Poisson(λ) distribution. Consider the following two decision rules for estimating λ with squared error loss, based on $X = (X_1, X_2)$:

$$\begin{aligned} \delta_1(x) &= \frac{x_1 + x_2}{2} \\ \delta_2(x) &= \begin{cases} 1 & \text{if } x_1 = 0 \text{ or } x_2 = 0 \\ 2 & \text{otherwise} \end{cases} \end{aligned}$$

1. Show that δ_2 does not dominate δ_1.
2. Show that δ_1 does not dominate δ_2.
3. Use the Rao-Blackwell theorem to find a decision rule that dominates δ_2. Give an explicit formula for this decision rule, and find a value of λ for which it has lower risk than δ_2.

Q2: Data $X_1, \ldots, X_n \in (0, 1)$ are modelled using a parameter $\theta \in (0, 1)$, with $X_1, \ldots, X_n$ IID given θ, each with density

$$f_{X_i|\theta}(x|\theta) = \begin{cases} (1-\theta)/2 & \text{if } x \le 1/2 \\ \theta/2 & \text{if } x > 1/2 \end{cases}$$

We wish to estimate θ with squared error loss. The estimator $\delta_0(x) = (1/n) \sum x_i$ has been proposed. Using the Rao-Blackwell theorem, find a better estimator.

Q3: Given $\theta \in R$, data $X_1, \ldots, X_n$ are IID distributed according to the Exp(1) distribution shifted right by θ — ie, we can write $X_i = Z_i + \theta$ with $Z_i \sim \text{Exp}(1)$. We wish to estimate θ, with squared-error loss. One possible estimator is $\delta_0(x) = \overline{x} - 1$, which is unbiased, and equivariant.

1. Find the risk function for δ_0.
2. Find the minimal sufficient statistic for this model, and then by applying the Rao-Blackwell theorem, find an estimator, δ_1, that should have risk at least as small as δ_0.
3. Find the risk function for δ_1.
4. Find the Pitman estimator, δ_2, of θ starting with δ_0. Comment on its relationship with δ_1.