Name:

STA 3000 — Test #1 — 2009-10-30

For all questions, show enough of your work to indicate how you obtained your answer. No books or notes are allowed.

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1. Given $\theta \in (0, \infty)$, observations X and Y are independent, with X having the Poisson(θ) distribution and Y having the Poisson($1/\theta$) distribution. The Poisson probability function (over the non-negative integers) is $f(z; \lambda) = \lambda^z \exp(-\lambda)/z!$.

   Find a simple form of the minimal sufficient statistic for this model, and prove it is minimal sufficient.

2. Given $\theta \in (1, \infty)$, observations $Y_1, \ldots, Y_n \overset{\text{IID}}{\sim} U(0, \theta)$. Suppose that we use as the prior for θ the distribution with density proportional to $I(\theta > 1)/\theta^2$.

   (a) Find the posterior density for θ given $y_1, \ldots, y_n$.

(b) Find the predictive density for a new observation, Y_{n+1}, given $y_1, \ldots, y_n$. Assume that, given a value for θ, Y_{n+1} is independent of $Y_1, \ldots, Y_n$, and has the same distribution as each of $Y_1, \ldots, Y_n$.

3. Given $\theta \in (0, \infty)$, suppose $Y_1, \ldots, Y_n$ are IID observations with an exponential distribution with mean θ shifted to the right by θ. In other words, the density for each Y_i is $I(y > \theta)(1/\theta)\exp(-(y-\theta)/\theta)$.

   (a) Find a simple expression for the minimal sufficient statistic for this model and prove that it is minimal sufficient.

   (b) Show that the minimal sufficient statistic is not complete by showing that two different unbiased estimators of θ exist.

   (c) Find an ancillary statistic that is a function of the minimal sufficient statistic.

4. Consider the distribution of random variables $Y_1, Y_2, Y_3, \ldots$ in $\{0, 1\}$ defined by

$$P(Y_{i+1} = 1 \mid Y_1 = y_1, \ldots, Y_i = y_i) \;=\; \Big(2 + \sum_{j=1}^{i} y_j\Big) \;\Big/\; (i+3)$$

For $i = 0$, this should be interpreted as $P(Y_1 = 1) \;=\; 2/3$.

(a) Prove that the distribution of $Y_1, Y_2, Y_3, \ldots$ is exchangeable.

(b) Find the mean and variance of $\overline{Y}_n = (Y_1 + Y_2 + \cdots + Y_n)/n$ (directly, without using DeFinetti's Represenation Theorem).

(c) Comment on the meaning of the limit of the mean and variance of $\overline{Y}_n$ found in part (b) as n goes to infinity, with reference to DeFinetti's Represenation Theorem.