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STA 3000 — Test #2 — 2009-12-11

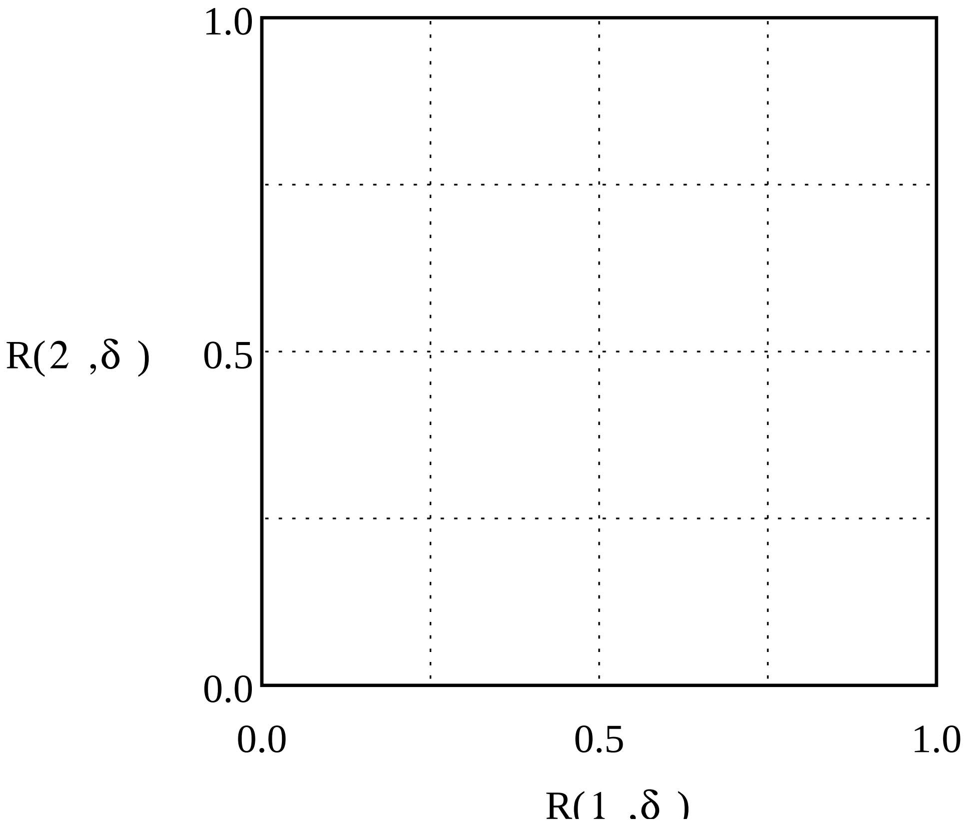
For all questions, show enough of your work to indicate how you obtained your answer. No books or notes are allowed.

1. Consider two pairs of data points, (X_1, Y_1) and (X_2, Y_2), with $(X_i, Y_i) \in \{1, 2, 3\}^2$. We model these pairs using a parameter $\theta \in \{1, 2\}$. Given θ, the two pairs are independent. If $\theta = 2$, the distribution of each is uniform over the data space, and if $\theta = 1$, the distribution of each is an equal mixture of this uniform distribution and the uniform distribution over the subset where $X_i = Y_i$. In other words,

$$(X_1, Y_1),\ (X_2, Y_2) \mid \theta \overset{\text{IID}}{\sim} (\theta/2)\,\text{Uniform}(\{1, 2, 3\}^2) \ +\ (1 - \theta/2)\,\text{Uniform}(\{(1,1),\ (2,2),\ (3,3)\})$$

We wish to estimate θ from the observed (x_1, y_1) and (x_2, y_2), with loss function $L(\theta, a) = |\theta - a|$, where a is in the action space $\{1, 2\}$.

   a) On the diagram on the next page, plot the risk set for this decision problem — ie, the set of risk functions that can be obtained from a randomized or non-randomized decision rule. Explain how this can be done without explicitly considering all $2^{81} = 2417851639229258349412352$ non-randomized decision rules.


1.0
0.5
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R(2 ,δ)
0.0
0.5
1.0
R(1 ,δ)

b) Find two non-randomized Bayes rules for this decision problem that are admissible, and that have different risk functions.

c) Are any of the Bayes rules for this decision problem inadmissible? If so, give one.

d) Find a minimax rule for this decision problem (either randomized or non-randomized), and find a prior distribution for θ for which this minimax rule is a Bayes rule.

2. $X_1, \ldots, X_n$ are non-negative integers that are IID from the Poisson distribution with mean λ, which has probability mass function $(\lambda^x/x!)\exp(-\lambda)$. We do not necessarily observe $X_1, \ldots, X_n$, however. Instead, we observe $Y_1, \ldots, Y_n$, in which (independently for each i) Y_i is X_i with probability p, and otherwise Y_i is -1. (You can think of this as a model with some "missing data".)

   a) Show that this model for $Y_1, \ldots, Y_n$ (with parameters $p \in (0, 1)$ and $\lambda \in (0, \infty)$) is an exponential family model.

b) Find the expectation of the natural sufficient statistics for your formulation of this model, by differentiating the normalizing factor of the distribution (*not* by some other method).

3. Given λ, observations $y_1, \ldots, y_n$ are IID from the Poisson distribution with mean λ. We wish to estimate $\exp(-\lambda)$ with squared error loss — ie, our loss function is $L(\lambda, a) = (\exp(-\lambda) - a)^2$.

   Consider the following estimator:

$$\delta(y) \;=\; \frac{1}{n+1}\Big(1 \;+\; \sum_{i=1}^{n} I(y_i = 0)\Big)$$

   Find a better estimator using the Rao-Blackwell theorem.